Representation of the Broad-Scale Spectral Form in the Two-Scale Approximation for the Full Boltzmann Integral

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Nonlinear wave-wave interactions are a primary controlling mechanism in wave generation. The Two-Scale-Approximate (TSA) could help overcome existing problems with the representation of this term; however, the transition of this approximation from a theoretical construct to an operational tool is still a “work in progress.”

A fundamental question regarding the “operationalization” of the TSA is “how will certain pre-calculated matrices be handled and how much storage will be required to handle them?”

This paper will address these questions.
APPROACH:

• Review the form of the TSA to determine how the formulation is posed in terms of the 2 interacting scales

• Examine the variability of observed directional spectra for the “simple” case of single-peaked wind seas

• Examine the ability of the 2nd scale within the TSA to capture the effects of deviations between the broad-scale spectral form and an actual spectrum

• Examine the effect of superposed swell wave trains upon the total wave-wave interactions

• Evaluate the relative merit of the TSA in terms of a source term for operational models
• Directional spectra in coastal and offshore areas appear to follow consistent similarity forms of the type first hypothesized by Kitaigorodskii (1962) - even in finite-depth situations.

• For single peaked wind-sea spectra, two parameters appear sufficient to provide a “good” representation of the nonlinear interactions, spectral peakedness and relative depth.

• Storage requirements for pre-calculated terms and coefficients appear to be very manageable.

• The relaxation of a spectral perturbation estimated by the TSA is very close to that of the FBI

• Since wave generation in many situations follows a similarity basis (Badulin et al, 2008), the TSA offers an very good basis for a highly accurate representation of “typical” wave generation cases.

• Interactions with swell could be added for a small range of significance.
RECENT PROGRESS IN TSA

Timing studies – TSA faster than DIA

TSA moved from FBI to separate code
• transfer matrices developed
• scaling (finally) complete
• Initial 2 papers published JPO
  - Theory and hypothetical spectra
  - Real spectra
• next manuscript on spectral evolution (here)
Basis for Two-Scale Approximation: Decompose spectrum into 2 parts – a Broad-Scale and a Perturbation Scale

\[ N^3 = \hat{n}_1\hat{n}_3(\hat{n}_4 - \hat{n}_2) + \hat{n}_2\hat{n}_4(\hat{n}_3 - \hat{n}_1) + \]
\[ n'_1n'_3(n'_4 - n'_2) + n'_2n'_4(n'_3 - n'_1) + \]
\[ \hat{n}_1\hat{n}_3(n'_4 - n'_2) + \hat{n}_2\hat{n}_4(n'_3 - n'_1) + \]
\[ n'_1n'_3(\hat{n}_4 - \hat{n}_2) + n'_2n'_4(\hat{n}_3 - \hat{n}_1) + \]
\[ \hat{n}_1n'_3(\hat{n}_4 - \hat{n}_2) + \hat{n}_2n'_4(\hat{n}_3 - \hat{n}_1) + \]
\[ n'_1\hat{n}_3(\hat{n}_4 - \hat{n}_2) + n'_2\hat{n}_4(\hat{n}_3 - \hat{n}_1) + \]
\[ \hat{n}_1n'_3(n'_4 - n'_2) + \hat{n}_2n'_4(n'_3 - n'_1) + \]
\[ n'_1\hat{n}_3(n'_4 - n'_2) + n'_2\hat{n}_4(n'_3 - n'_1) \]

\[ n = \hat{n} + n' \]

\[ S_{nl}(f, \theta) = B + L + X \]

Line 1 contains interactions for only B
Line 2 contains interactions for only L
Lines 3-8 contain cross-interactions between B and L
This approximation to the full integral would be exact if all terms were retained.

\[
\frac{\partial n_1}{\partial t} = \int \int \int N_A^3 C \left| \frac{\partial W}{\partial n} \right|^{-1} ds \ k_3 d\theta_3 dk_3 \\
+ \int \int \int N_B^3 C \left| \frac{\partial W}{\partial n} \right|^{-1} ds \ k_3 d\theta_3 dk_3 \\
+ \int \int \int N_C^3 C \left| \frac{\partial W}{\partial n} \right|^{-1} ds \ k_3 d\theta_3 dk_3
\]
The fundamental idea here is to capture the broad-scale distribution of energy parametrically and to allow “local” differences to be treated as shown below. Terms that are neglected tend to contribute in a +/- sense around locus – “s.”

This could be a DIA form or a diffusion operator, but we would lose considerable accuracy.

\[ \frac{\partial n_1}{\partial t} = B + \iiint N_3^3 C \left| \frac{\partial W}{\partial n} \right|^{-1} ds k_3 d\theta_3 dk_3 + \ldots \]

\[ N_3^3 \text{ terms neglect terms containing } n'_2 \text{ and } n'_4 \text{ - retain } \hat{n}_2 \text{ and } \hat{n}_4 \]

\[ N_3^3 = \hat{n}_2 \hat{n}_4 (n'_3 - n'_1) + n'_1 n'_3 (n'_4 - \hat{n}_2) + \hat{n}_1 n'_3 (\hat{n}_4 - \hat{n}_2) + n'_1 \hat{n}_3 (\hat{n}_4 - \hat{n}_2) \]

Note: X is typically 2-3 times larger than L or B. This is why linear sums (neural networks, EOF’s, etc.) do not work well for Snl estimation.
\[
\frac{\partial n_1}{\partial t} = B + \left( \frac{k}{k_0} \right)^{19/2} \left( \frac{\beta}{\beta_0} \right) \iint (\hat{n}_1 n'_3 + n'_1 \hat{n}_3 + n'_1 n'_3) \Lambda(\hat{n}_2 - \hat{n}_4, k_1, k_*, \theta_*, x_1, \ldots, x_n) \, k_*)d\theta_*dk_* 
+ \left( \frac{\beta}{\beta_0} \right)^2 \iint (n'_1 - n'_3) \Lambda(\hat{n}_2 \hat{n}_4, k_1, k_*, \theta_*, x_1, \ldots, x_n) \, k_*)d\theta_*dk_*
\]

where

\[\Lambda(\hat{n}_2 - \hat{n}_4, k_1, k_*, \theta_*, x_1, \ldots, x_n) = \int C \left| \frac{\partial W}{\partial n} \right|^{-1} (\hat{n}_4 - \hat{n}_2) \, ds\]

\[\Lambda(\hat{n}_2 \hat{n}_4, k_1, k_*, \theta_*, x_1, \ldots, x_n) = \int C \left| \frac{\partial W}{\partial n} \right|^{-1} \hat{n}_2 \hat{n}_4 \, ds\]

Note: These terms are pre-calculated. This removes all calculations from innermost loop in the integration.

\[\frac{\partial n_1}{\partial t} = \sum_{\delta f_i} \sum_{\delta \theta_j} \mu_{ij\delta p} N_{p*}^3 + \mu_{ij\delta d} N_{d*}^3\]

Note: This has no free parameters. All coefficients are determined from the full integral. Still conserves action, energy, and momentum since each elemental transfer is conservative.

Pumping

Diffusion
JONSWAP (f^5 spectrum)

\[ E(f) = \frac{\alpha g^2}{(2\pi)^4} f^{-5} \exp \left[ -1.25 \left( \frac{f}{f_p} \right)^4 \right] \gamma_5^{\Theta_5} \]

where \( E(f) \) is the spectral energy density at \( f \),

\[ \Theta_5 = \exp \left[ \frac{-(f - f_p)^2}{2\sigma^2 f_p^2} \right] \]

and

\[ \sigma = \sigma_a \text{ for } f < f_p \]
\[ = \sigma_b \text{ for } f \leq f_p \]

Modified f^4 spectrum after Resio and Perrie (1989)

\[ E(f) = \frac{2\beta g}{(2\pi)^3} f^{-4} \left[ z_4 \left( \frac{f}{f_p} \right)^4 \exp(-\Theta_4) + 1 \right] \]

where

\( \beta \) is the equilibrium range constant as defined in Resio et al. (2004)

\( z_4 \) is a constant = \( \gamma_r \) for \( f \leq f_p \); \( \gamma_r - 1 \) for \( f > f_p \)

\( \gamma_r \) is the relative peakedness as defined in Long and Resio (2007)

\( \Theta_4 \) is a peakedness factor given by

\[ \Theta_4 = \left[ \frac{(f - f_p)}{2\sigma f_p} \right]^2 \]

with values the same as for the JONSWAP spectrum.
Idealized form for $f^4$ spectrum.

$Y_r$ = relative peakedness in $f^4$ spectrum
Spectra from around the world have shown a pronounced $f^{-4}$ ($k^{-3}$) form with a transition to $f^{-5}$ form at high frequencies.

\[ E_c(f) \sim E(f)f^4 \]

\[ \rightarrow F_c(k) \sim F(k)k^{-5/2} \]
Note f⁻⁵ form here
Analyses From Long And Resio 2007 JGR
Compensated Spectra from Hurricane Ivan

Stn: 41010  WaveAge: $U_{10}/C_p: 0.9 - 1.1$ [TotObs: 133]

$E(f) \propto f^{-5}$

$E_{MAX} > 25 \text{ m}^2\cdot\text{s}^{-1}$
Figure 2. Observed relationship between spectral peakedness and inverse wave age (Long and Resio, 2007). Relationship is not as chaotic as JONSWAP data indicated.
Characteristics of directional distributions of energy:
1. “Young” waves are very bimodal
2. “Old” waves approach unimodal
3. Both distributions are similar to Hasselmann et al. (1980)

Low inverse wave age (old waves) (Long and Resio, 2007)

High inverse wave age (young waves) (Long and Resio, 2007)

Variation in “n” obtained when fitting a $\cos^{2n}$ function compared to the data from Hasselmann et al (1980)
Both $f_p$ and $\beta$ scale out of the TSA representation. $\beta$ scales very well across many sets of experimental data.

$r^2 = 0.939$

Sample data points include:
- Currituck Sound
- Lake George
- FRF Waverider, #630
- FRF Baylor, #625
- NDBC 41001
- NDBC 46035
Effectiveness of 2\textsuperscript{nd} term on improving the fit of full TSA over parametric term alone for case of observed bimodal directional distribution to $\cos^6$ distribution.

Currituck case from Long and Resio (2007).

Comparison of parametric term alone in TSA (green line) versus parametric plus second term in the TSA (blue line) to the full integral solution (red line).
Effectiveness of TSA for case of skewed directional distribution to $\cos^6$ distribution.

Contours of directional spectral energies from depth of 17 m offshore of Duck, NC

Comparison of TSA solution and DIA for skewed directional distribution.
Some Additional Views: Compensated Spectra

\[ E_c(f) \sim E(f) f^4 \rightarrow F_c(k) \sim F(k) k^{-5/2} \]
Compensated low-peakedness spectra from waverider in 17 meter depth off of Duck, NC
Compensated high-peakedness spectra from waverider in 17 meter depth off of Duck, NC
What about Mixed Sea and Swell??

Two cases plot essentially over the no swell line.

Sea spectrum with 3 swell spectra having different separations

SNL for sea spectrum with 3 swell spectra having different separations
Varying swell steepness

SNL for varying swell steepness
Some Additional Views: Coastal Mixed Sea and Swell Directionally Aligned
FRF Waverider at 0210140100

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<th>$H_{m0}$ (m)</th>
<th>$f_{p,ws}$ (Hz)</th>
<th>$\theta_p$ (deg)</th>
<th>$u_{10}$ (m/s)</th>
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FRF Waverider at 0210140700

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TSA and FBI have essentially the same relaxation rate to a perturbation.
Fetch-growth and associated spectra have long been recognized to follow a self-similar pattern.

So, the first scale of the TSA should be able to capture most of the details of SNL during this growth.

2\textsuperscript{nd} term in TSA can focus on more complex situations – which is more along the lines of Hasselmann’s concept for wave growth in 1976.
QUESTIONS???