Flux balance and self-similar laws of wind-wave growth

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Supported by the U.S. Army Corps of Engineers grants W912HZ-04-P-0172, DACW42-03-C-0019, ONR grant N00014-03-1-0648, INTAS grant 01-234, Russian Foundation for Basic Research N04-05-64784 and the Russian Academy Program `Nonlinear Dynamics'

The Hasselmann equation (kinetic equation for WW)

$$dn_k/dt = S_{nl} + S_{input} + S_{diss}$$

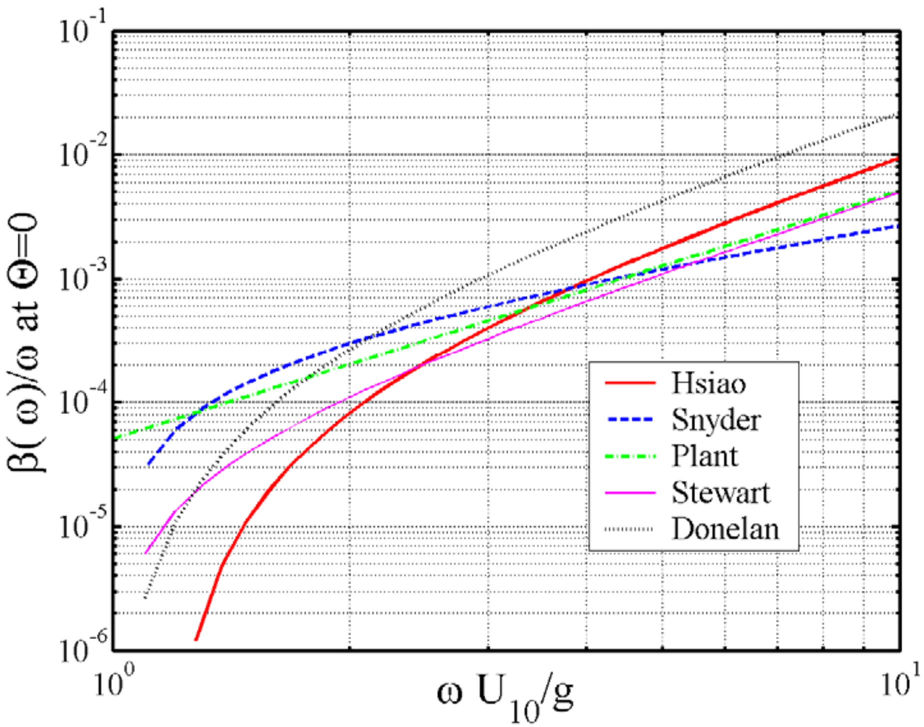
$$S_{nl} = 2\pi \int |T_{0123}|^2 (n_0 n_2 n_3 + n_1 n_2 n_3 - n_0 n_1 n_2 - n_0 n_1 n_3)$$
$$\times \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3)\delta(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3$$

S_{input} , S_{diss} - empirical parameterizations

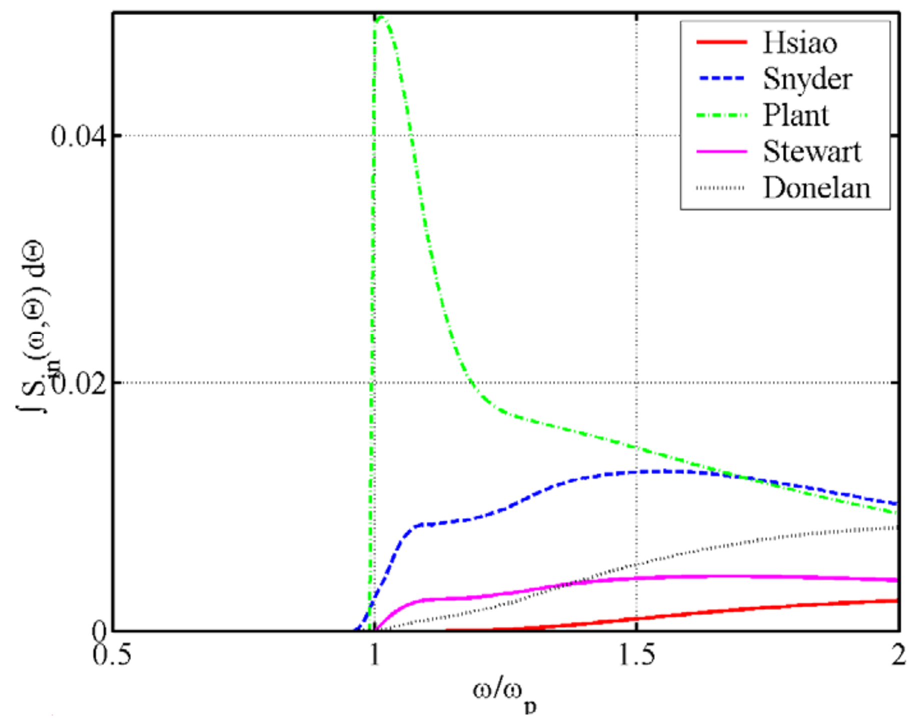
$N = \int n_k d\vec{k}$ - the wave action

$\varepsilon = \int \omega_k n_k d\vec{k}$, $\vec{P} = \int \vec{k} n_k dk$ - the energy and momentum

Non-dimensional wave input rates



Wave input term S_{in} for $U_{10}\omega_p/g=1$



What parameterization is true?

S_{diss} is known even worse than S_{in}

Important! For swell:

$$dN/dt = 0,\ d\varepsilon/dt < 0,\ dP/dt \neq 0$$

<u>Our key point</u>

Nonlinearity dominates !

$$S_{nl} >> S_{input}\ ,\ \ S_{diss}$$

There is no characteristic scale (deep water waves), i.e. homogeneity of the collision integral gives

$$S_{nl} \sim \lambda\, \mathbf{n}^{3}\, \mathbf{k}^{19/2}\ ,\ 1 < \lambda < 100$$

Why nonlinearity dominates?

P 1: *Badulin, Pushkarev, Resio, Zakharov:* Self-similarity of wind-driven seas. *Nonlinear Processes in Geophysics, **12**, 891-945, 2005*

P 2: *V. E. Zakharov: The Lord is graduated in theoretical physics*

One can develop an approximation procedure !!!

Mathematical jugglery or severe physics ?

We split wind-wave balance into two parts

$$\frac{dn_k}{dt} = S_{nl}$$

Conservative KE (solution shape)

$$\frac{d\langle n_k \rangle}{dt} = \langle S_{in} \rangle + \langle S_{diss} \rangle$$

Integral balance for the wind-driven waves

Does this model work?

How this model works?

Self-similar solutions (duration-limited) for power-law wave input (total flux)

$$n = at^{\alpha} U_{\beta}(b\mathbf{k}t^{\beta})$$

A family of SS solutions of the conservative KE

$$\int n d\mathbf{k} \sim ct^{r}$$

"Boundary condition" (integral balance) to select a solution

$$\alpha = \frac{19\beta - 2}{4} \qquad a = b^{19/4}$$

$$c = b^{11/4} \qquad r = \frac{11\beta - 2}{4}$$

For swell r=0, β=2/11

$$E(\omega, \vartheta, T) = C_p \frac{gP^{1/3}}{\omega^4}$$

Flux of energy $P \sim V^3$

For self-similar solutions if $r \neq 0$

$$C_p = C_p(\zeta) \qquad \zeta = bkt^{\beta}$$

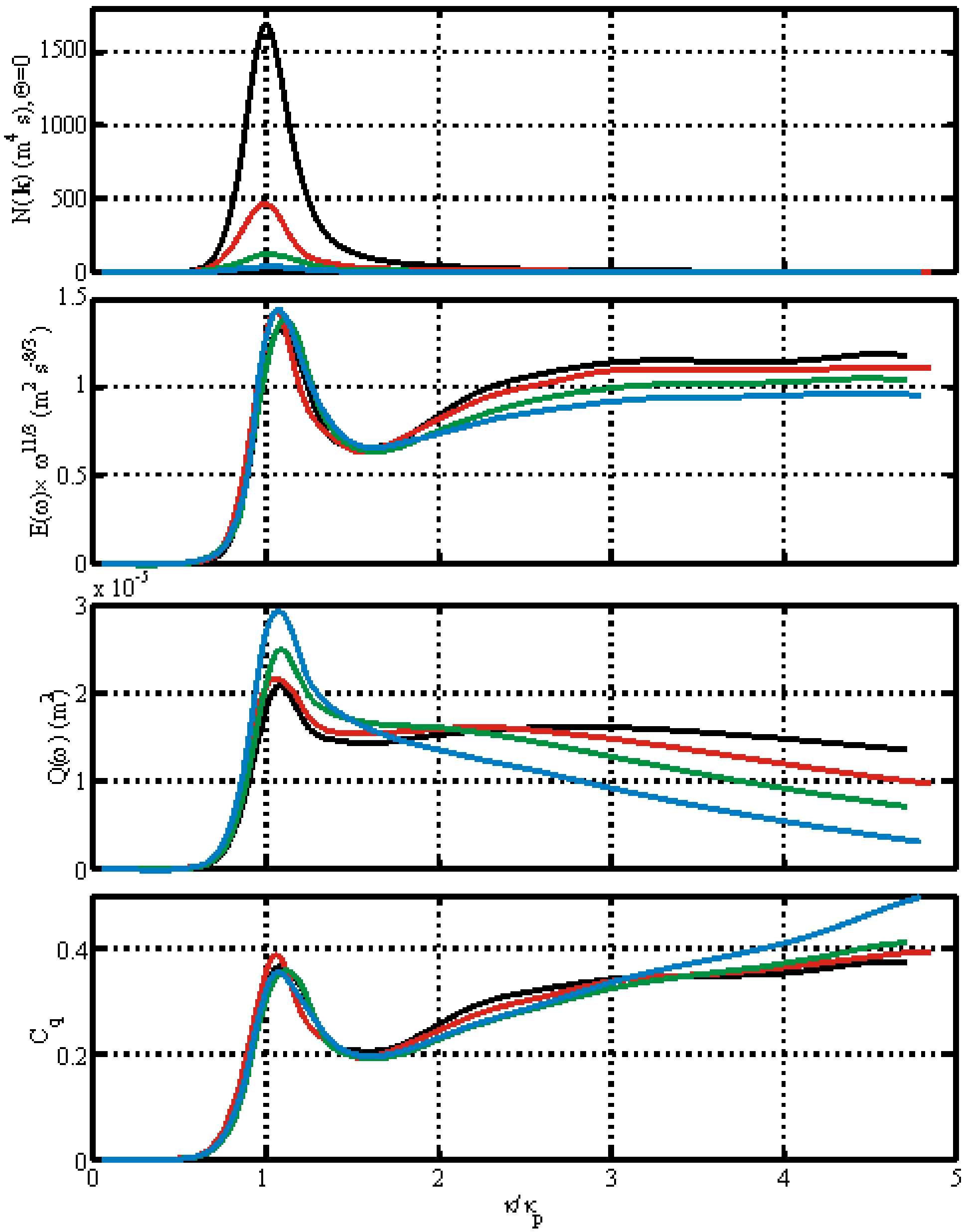
$$C_p \sim \zeta^{1/3} \quad \text{at} \quad \zeta \to \infty$$

$$\varepsilon(\omega) \sim C_q \frac{Q^{1/3}}{\omega^{11/3}}, \; Q$$ - flux of wave action

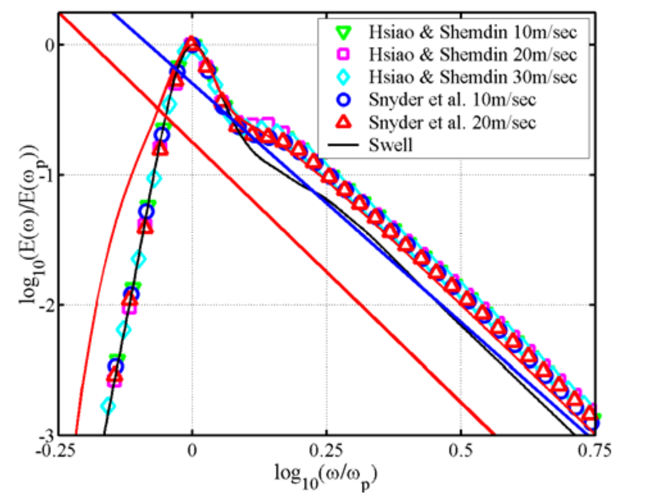
C_p, C_q - Kolmogorov constants

For $\varepsilon(\omega) \sim \omega^{-4}$, $\varepsilon(\omega) \sim \omega^{-11/3}$ $\quad S_{nl} = 0$

Spectra and fluxes vs time (numerical solutions of KE)



Spectra and fluxes for different models of wave input



Spectra and fluxes are quasi-universal, depend slightly on source function (on parameters of SS solutions)

Working within the split balance model we operate with "internal" scales of the conservative KE and with "external" scales of fluxes (net wave input)

NO WIND SCALING !!!

$$\frac{dn_k}{dt} = S_{nl}$$

Conservative KE (solution shape)

$$\frac{d\langle n_k \rangle}{dt} = \langle S_{in} \rangle + \langle S_{diss} \rangle$$

Integral balance for the wind-driven waves

"Correct" wind-wave spectrum (self-similarity in full)

$$\frac{E(\omega,\theta)\omega_p^5}{g^2} = \alpha^*_{ss}\left(\frac{\left\langle {dE}/{dt}\right\rangle \omega_p^3}{g^2}\right)^{1/3} \Phi_r\left(\frac{\omega}{\omega_p},\Theta\right)$$

ω_p – frequency scale (peak one is better than mean)

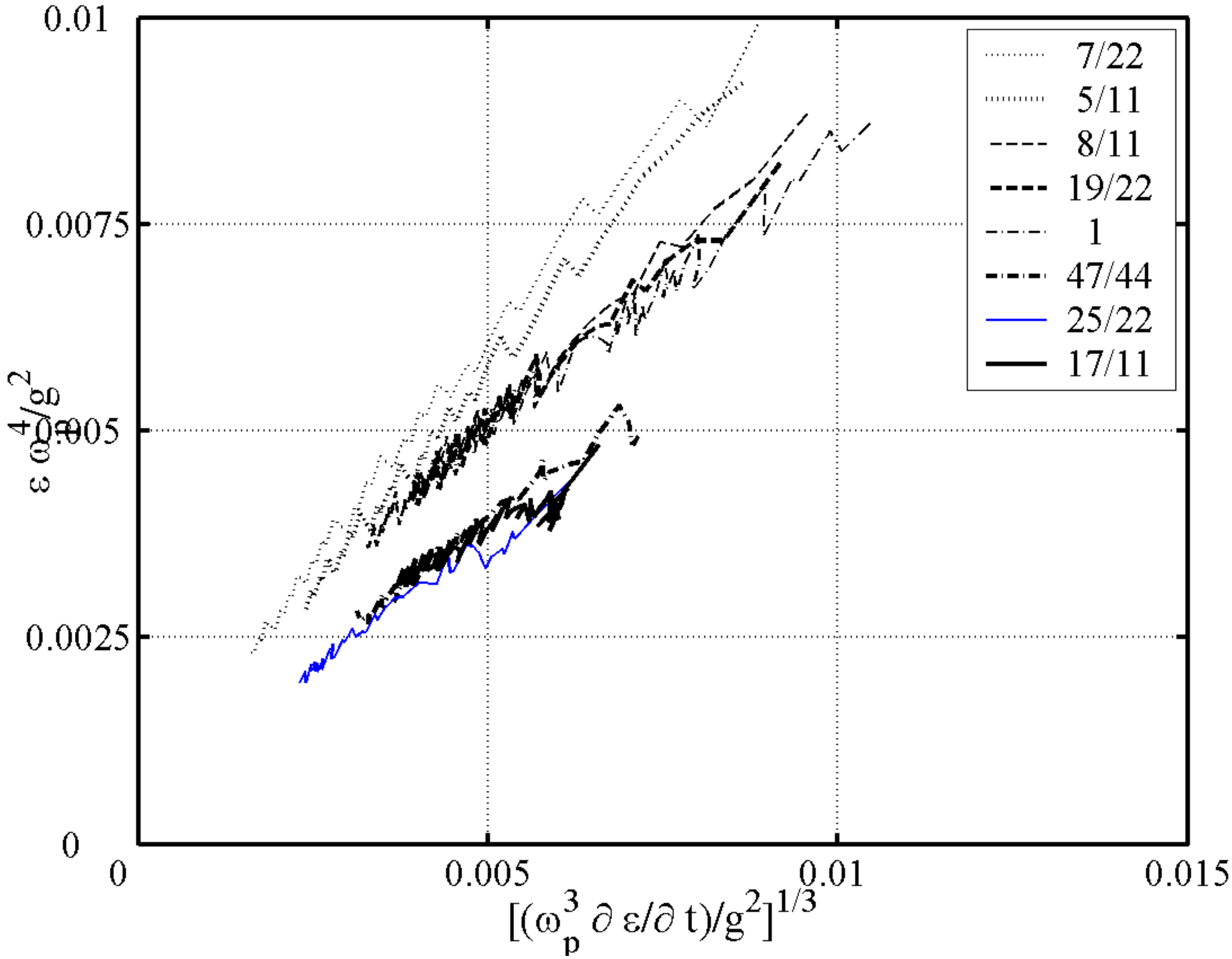
Φ_r – shape depends slightly on parameter ***r***

α^*_{ss} – self-similarity constant (of ***r***)

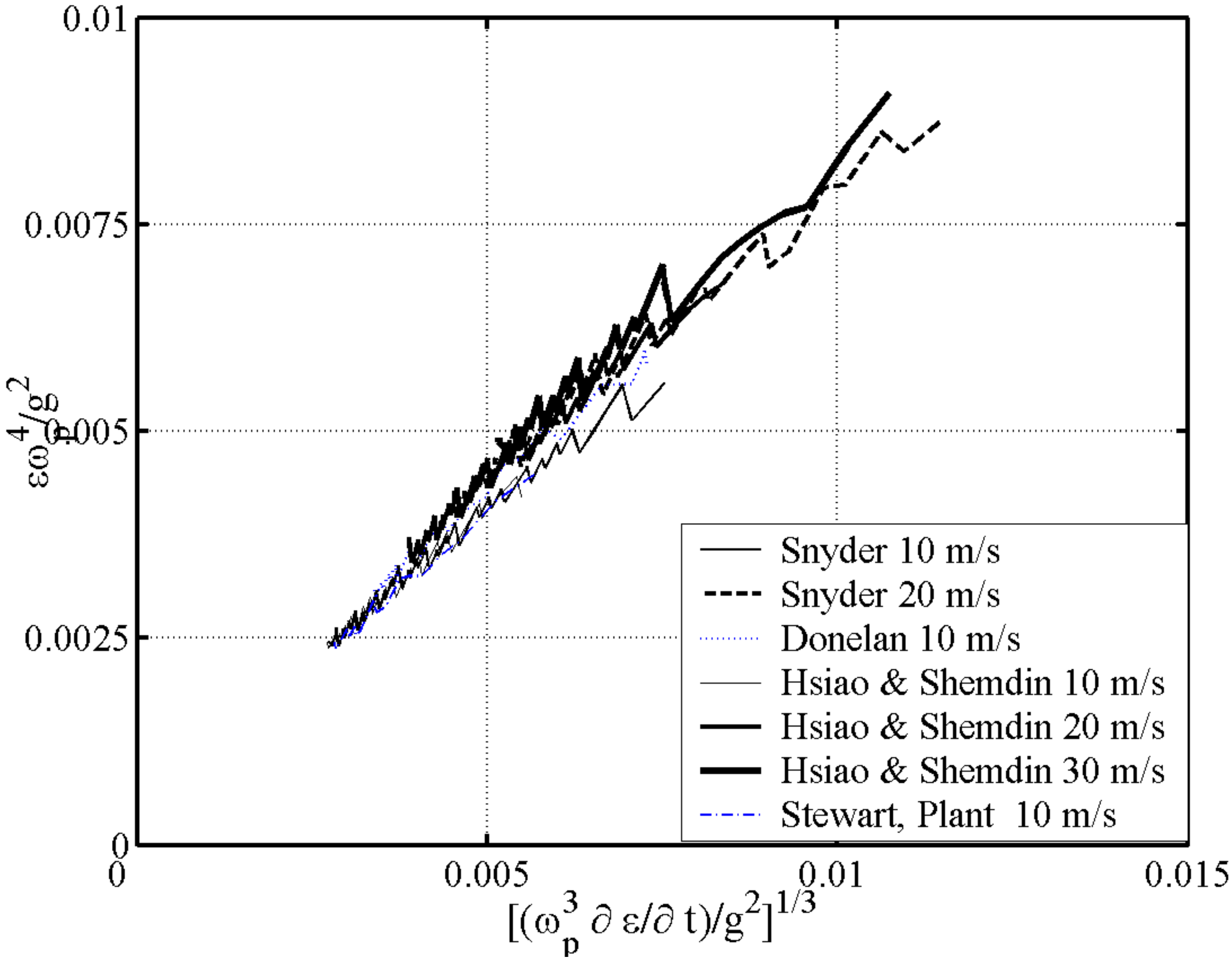
For total energy

$$\frac{\varepsilon\omega_p^4}{g^2} = \alpha_{ss}\left(\frac{{d\varepsilon}/{dt}\,\omega_p^3}{g^2}\right)^{1/3}$$

Check for numerical solutions (artificial wave pumping)



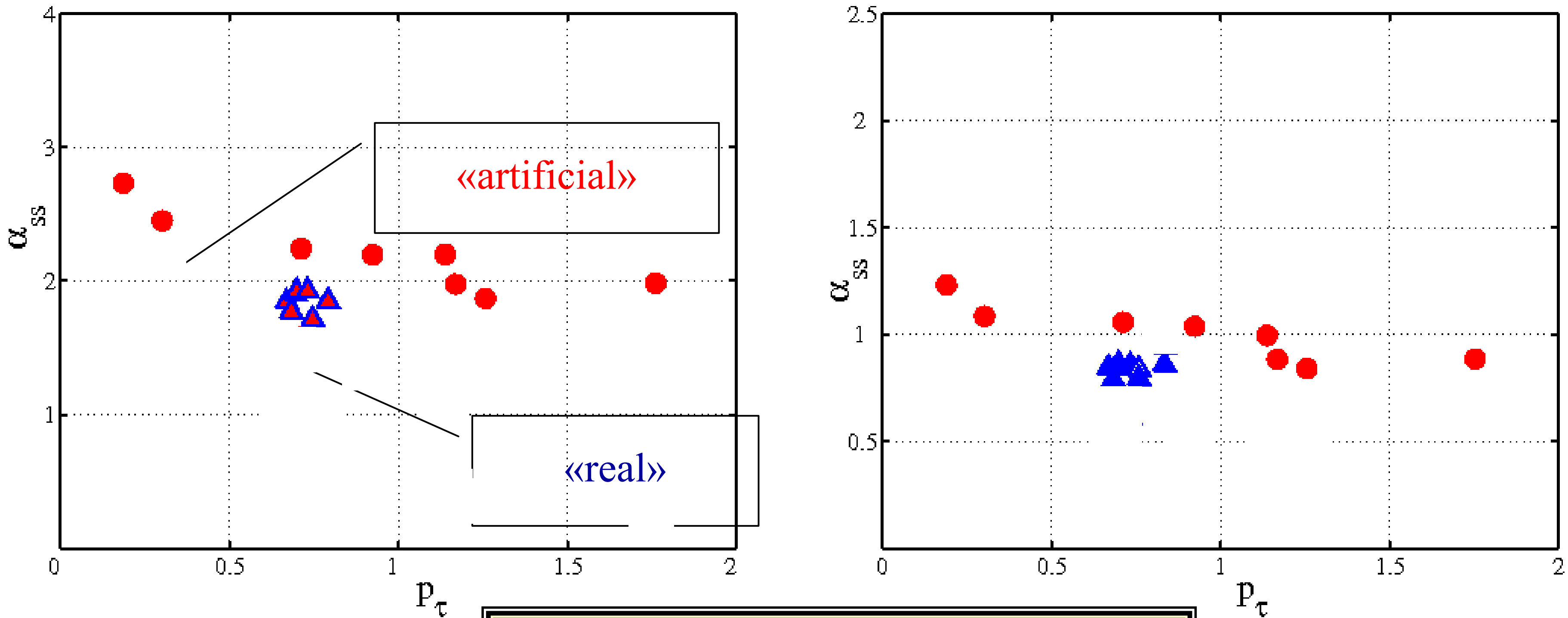
Check for numerical solutions ("real" wave pumping)



Self-similarity parameter α_{ss}

α_{ss} depends slightly on p !!!

(α_{ss} is an analogue of Kolmogorov's constants)



$$\frac{\varepsilon\omega_p^4}{g^2} = \alpha_{ss}\left(\frac{{}^{d\varepsilon}\!/\!{}_{dt}\,\omega_p^3}{g^2}\right)^{1/3}$$

Experimental dependences (wind speed scaling)

$$\widetilde{E} = \frac{Eg^2}{U_h^4}; \qquad \widetilde{\omega} = \frac{\omega U_h}{g}$$

- Duration-limited growth

$$\widetilde{E} = \widetilde{E}_0 \tau^{p_\tau};$$

$$\widetilde{\omega} = \widetilde{\omega}_0 \tau^{-q_\tau}$$

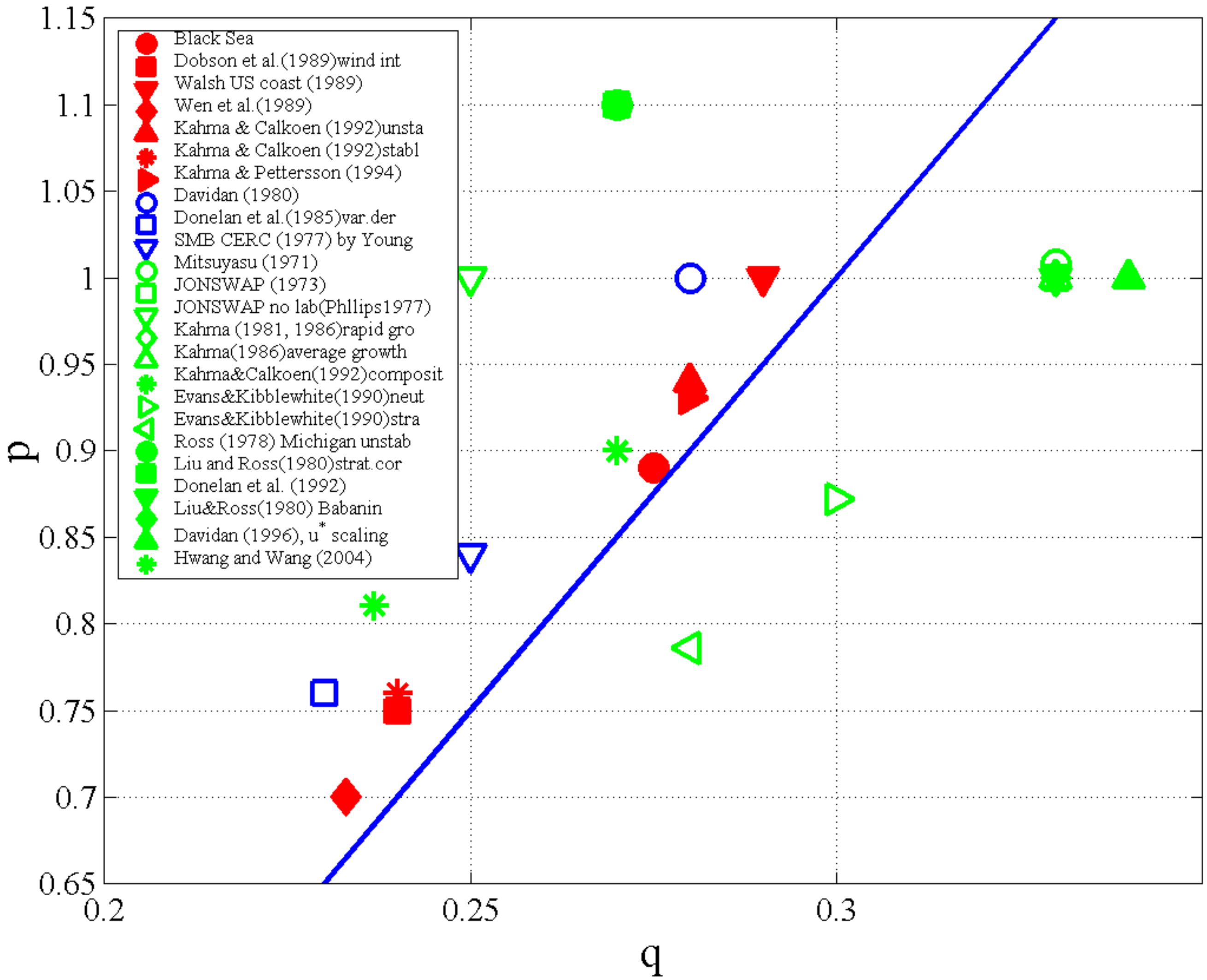
$$p_\tau = \frac{9q_\tau - 1}{2}$$

- Fetch-limited growth

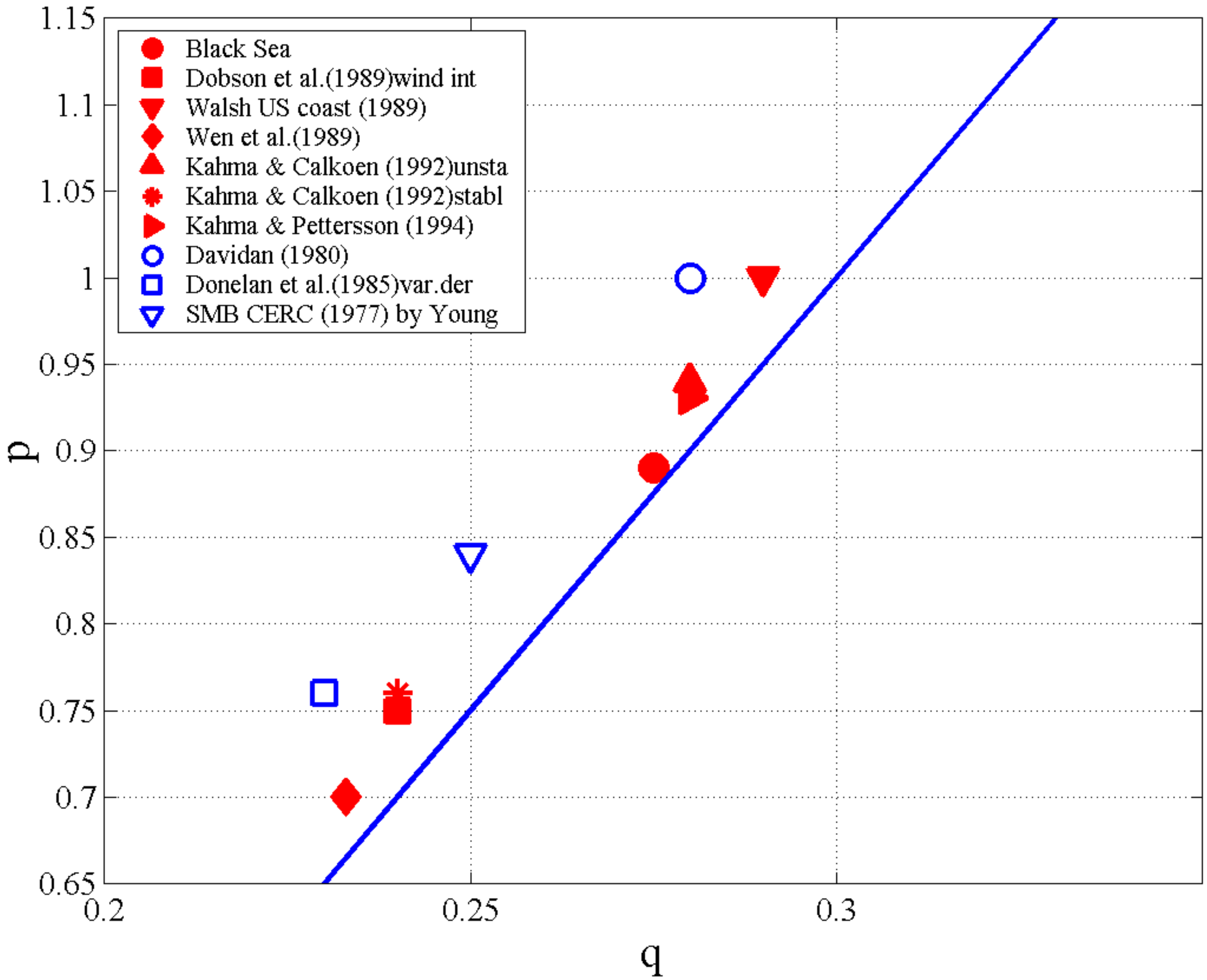
$$\widetilde{E} = \widetilde{E}_0 \chi^{p_\chi};$$

$$\widetilde{\omega} = \widetilde{\omega}_0 \chi^{-q_\chi}$$

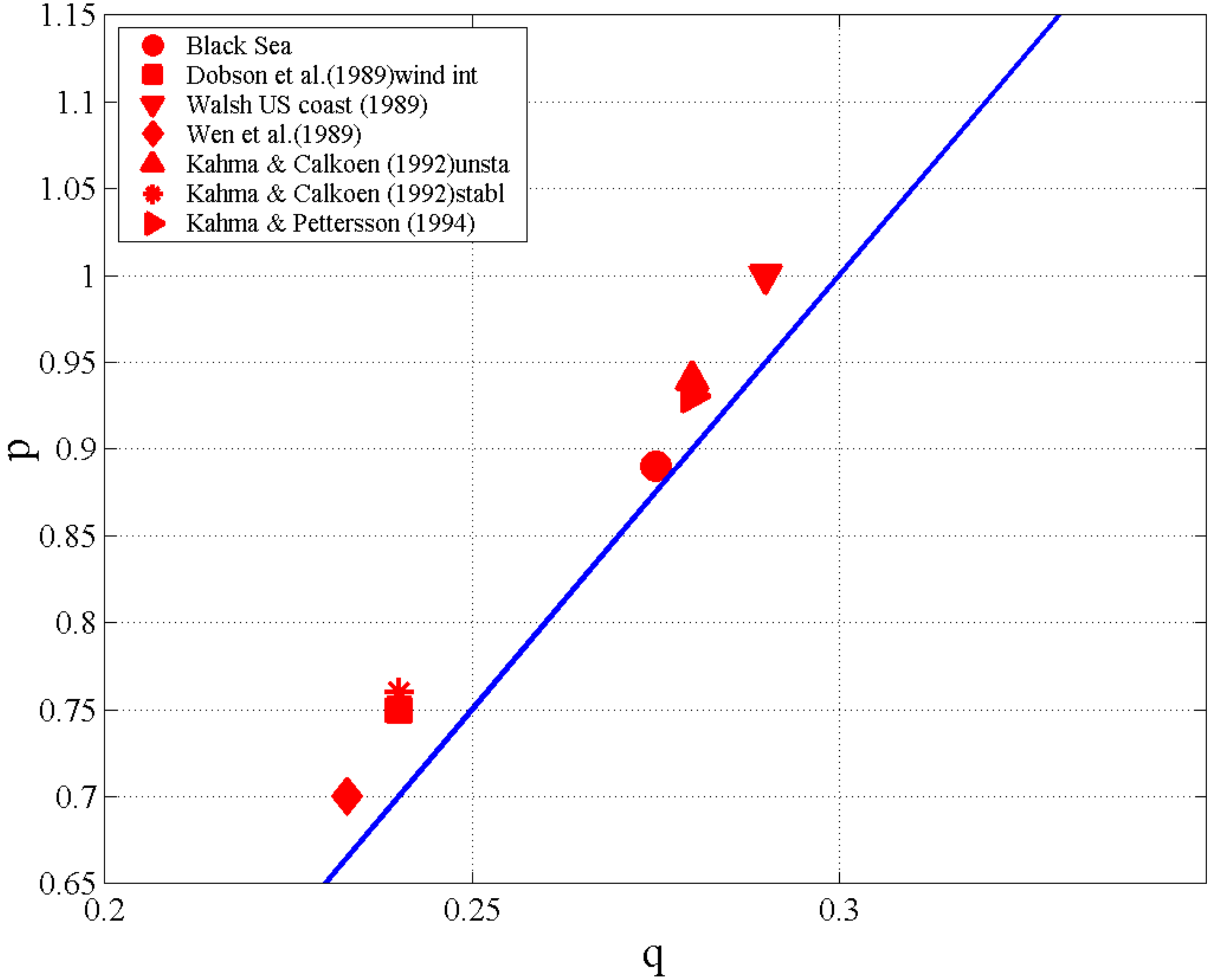
$$p_\chi = \frac{10q_\chi - 1}{2}$$



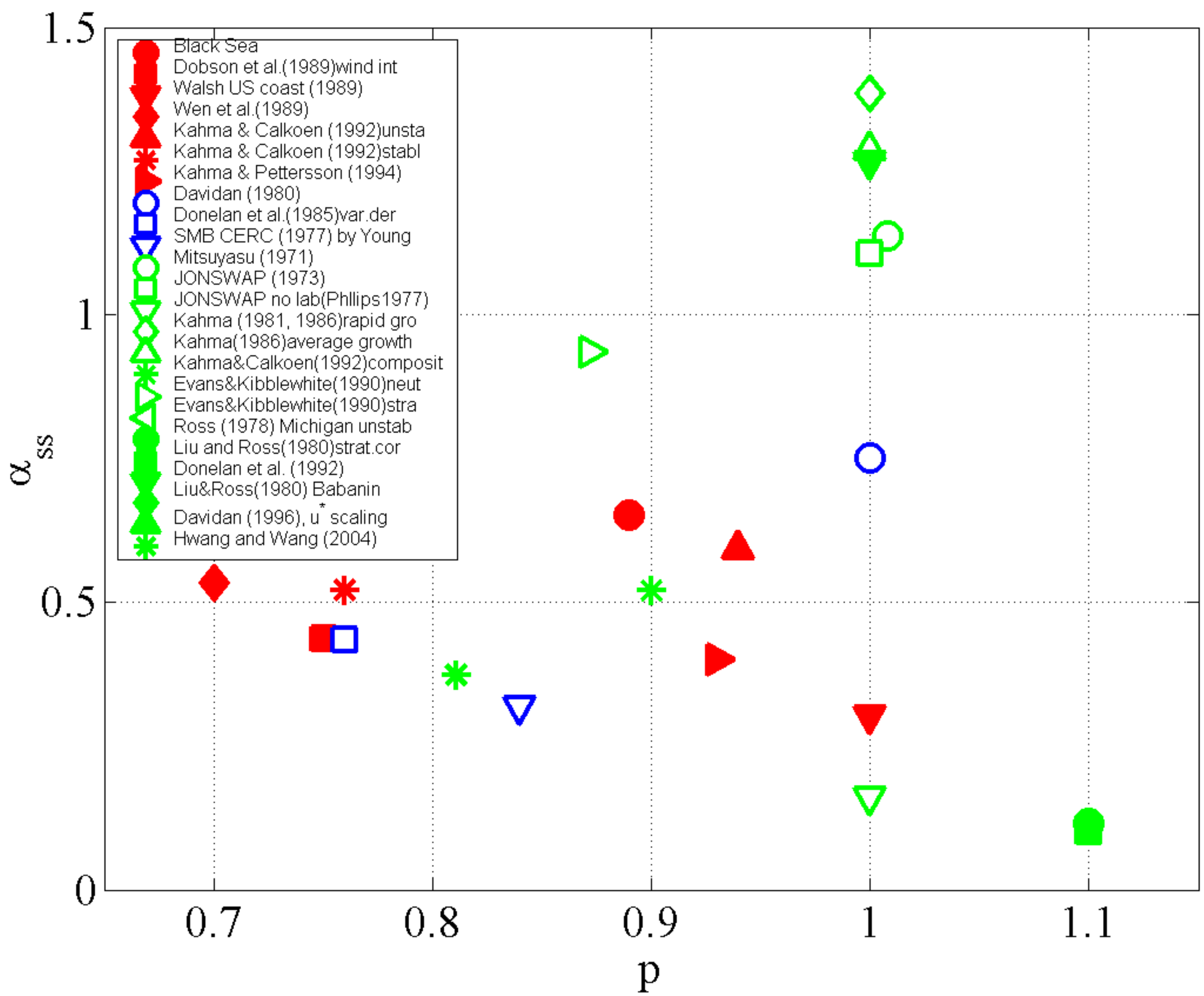
p(q) dependence for all three groups of 23 fetch-limited dependences



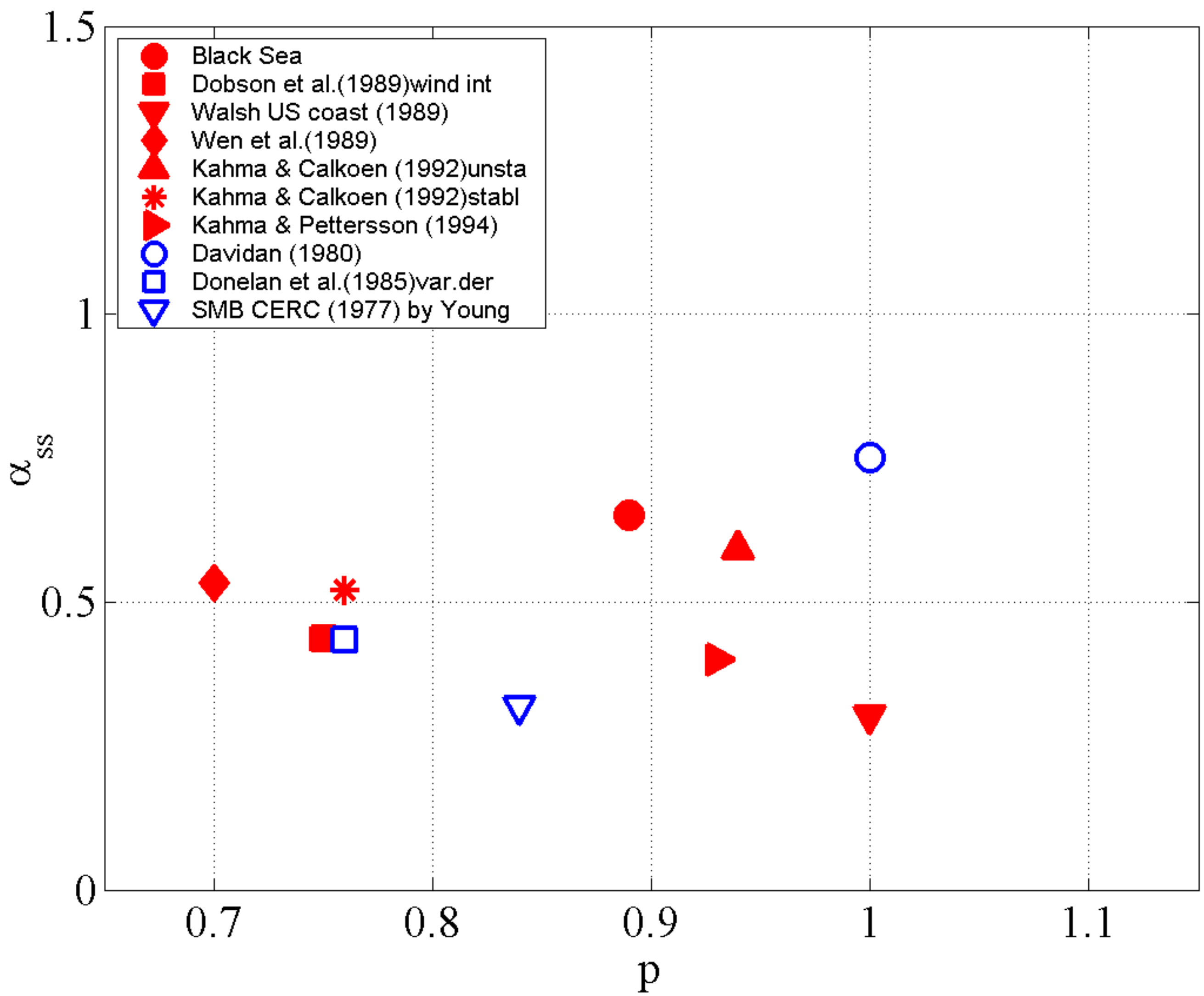
p(q) dependence for groups 1-2 of fetch-limited dependences (10 cases) – “good dependences”



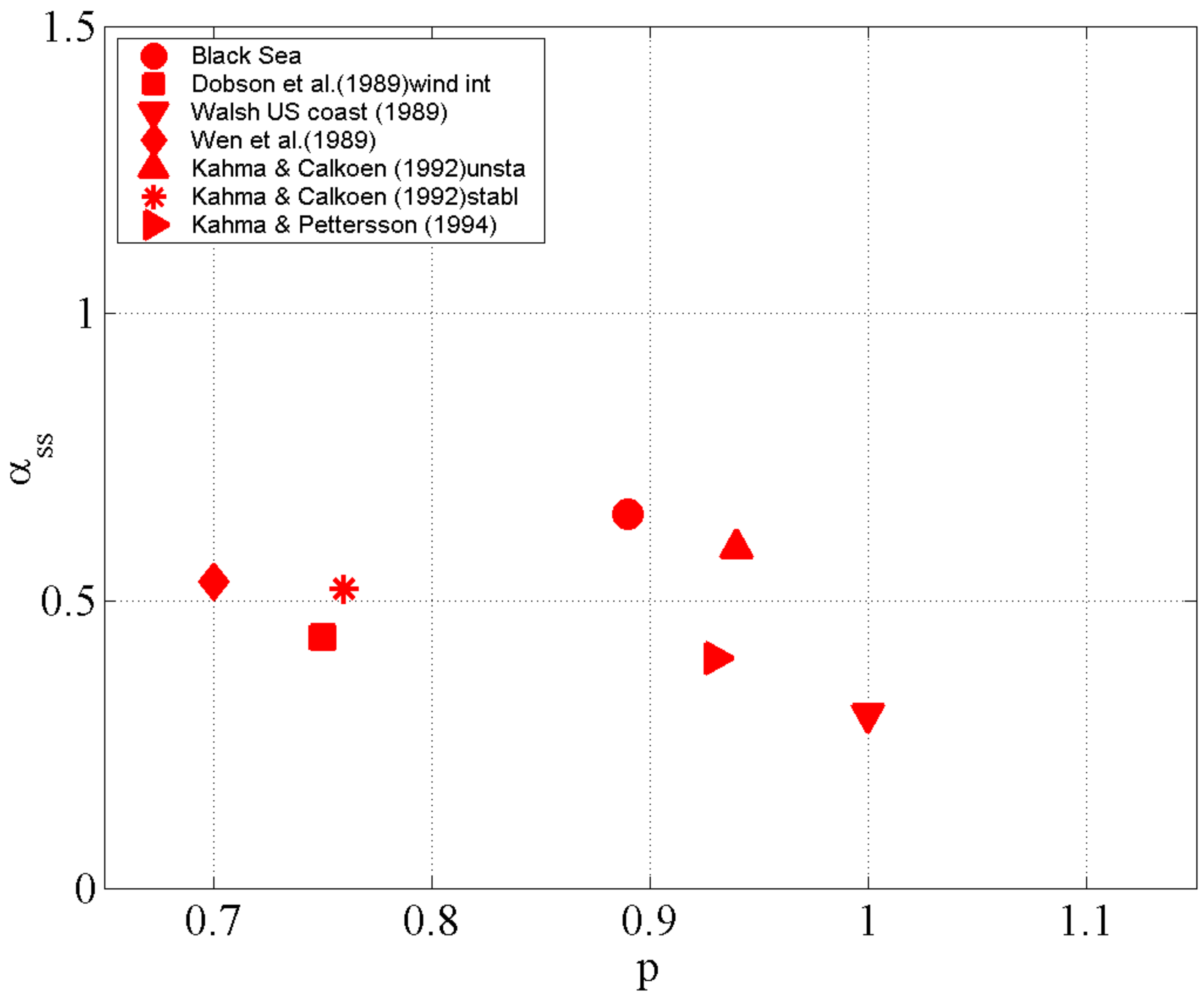
p(q) dependence for group 1 of fetch-limited dependences (7 “perfect cases”)



α(p) dependence for three groups of fetch-limited dependences



α(p) dependence for 1-2 groups of “good” fetch-limited dependences



α(p) dependence for group 1 of “perfect” fetch-limited dependences

Toba's law is a particular case of the weakly nonlinear law

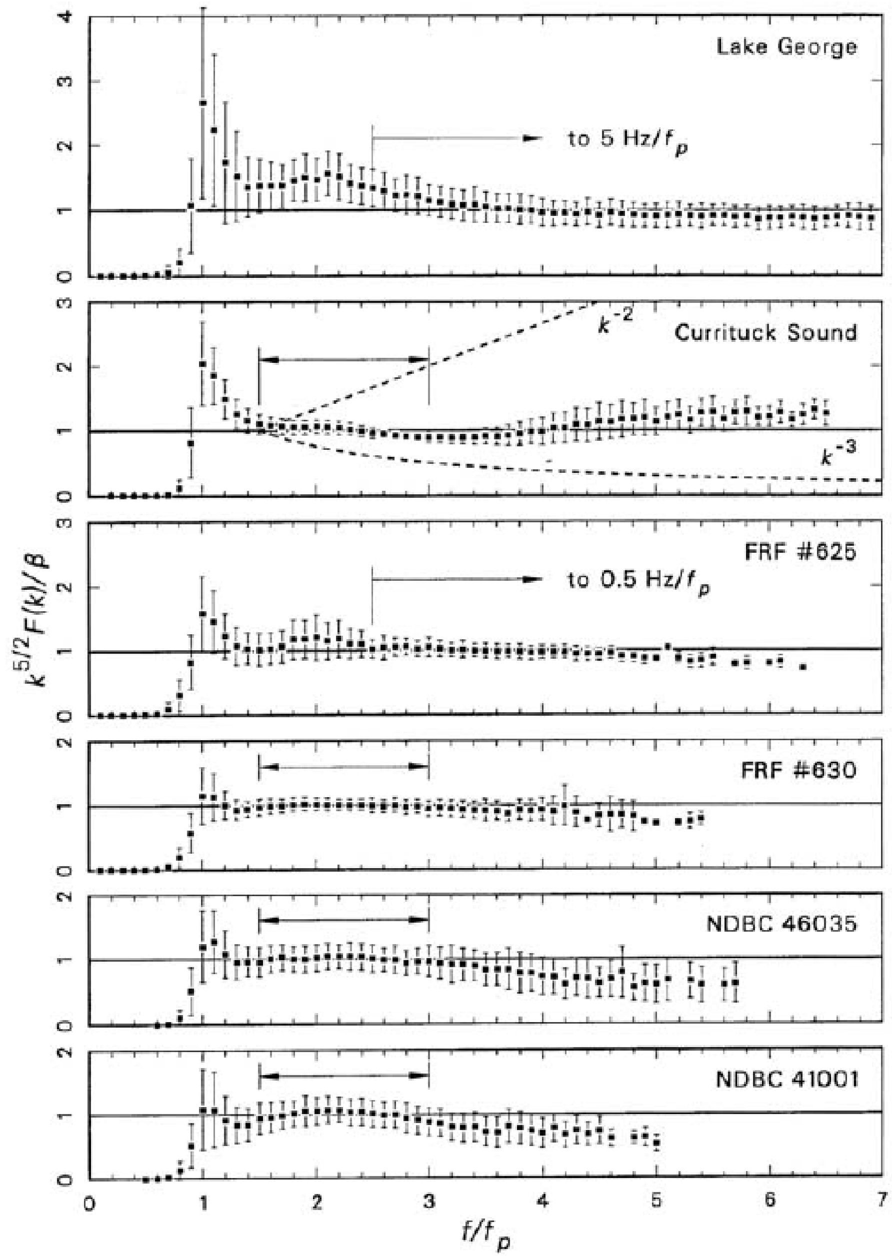
$$H_s = B(gu_*)^{1/2} T_s^{3/2}$$

- $d\varepsilon/dt$=const

$$d\varepsilon \big/ dt = \frac{\varepsilon^3 \omega_p^9}{\alpha_{ss}^3 g^4}$$

- For Toba's parameter $B=0.062$

$$d\varepsilon \big/ dt = 1.3 \frac{\rho_a}{\rho_w} \frac{u_*^3}{g}$$



Resio Long, JGR, 2004

(fig.3)

Energy flux in equilibrium range can be presented as

$P= C_{nl}\, g^{1/2}\, \beta^3$

Where

$\beta=1/2\ (u_a-u_0)\ g^{-1/2}$

And

$F(k)=\beta\, k^{-5/2}$

The link of spectra and fluxes is assumed in equilibrium range only

then

Parameterization of the link is found in terms of wind velocity from data analysis

Summary

- Nonlinear transfer dominates in KE = split balance

$\Downarrow$

- Self-similarity of KE solutions and fluxes
- Rigid bounds of the solutions and spectral fluxes

+

- Spectra and fluxes depend slightly on parameters of SS solutions (quasi-universality)

$\Downarrow$

Self-similar quasi-universal wave growth law

$\boldsymbol{E \sim P^{1/3}}$

Thanks for continuing

support to

US Army Corps of Engineers

and

Office of Naval Research