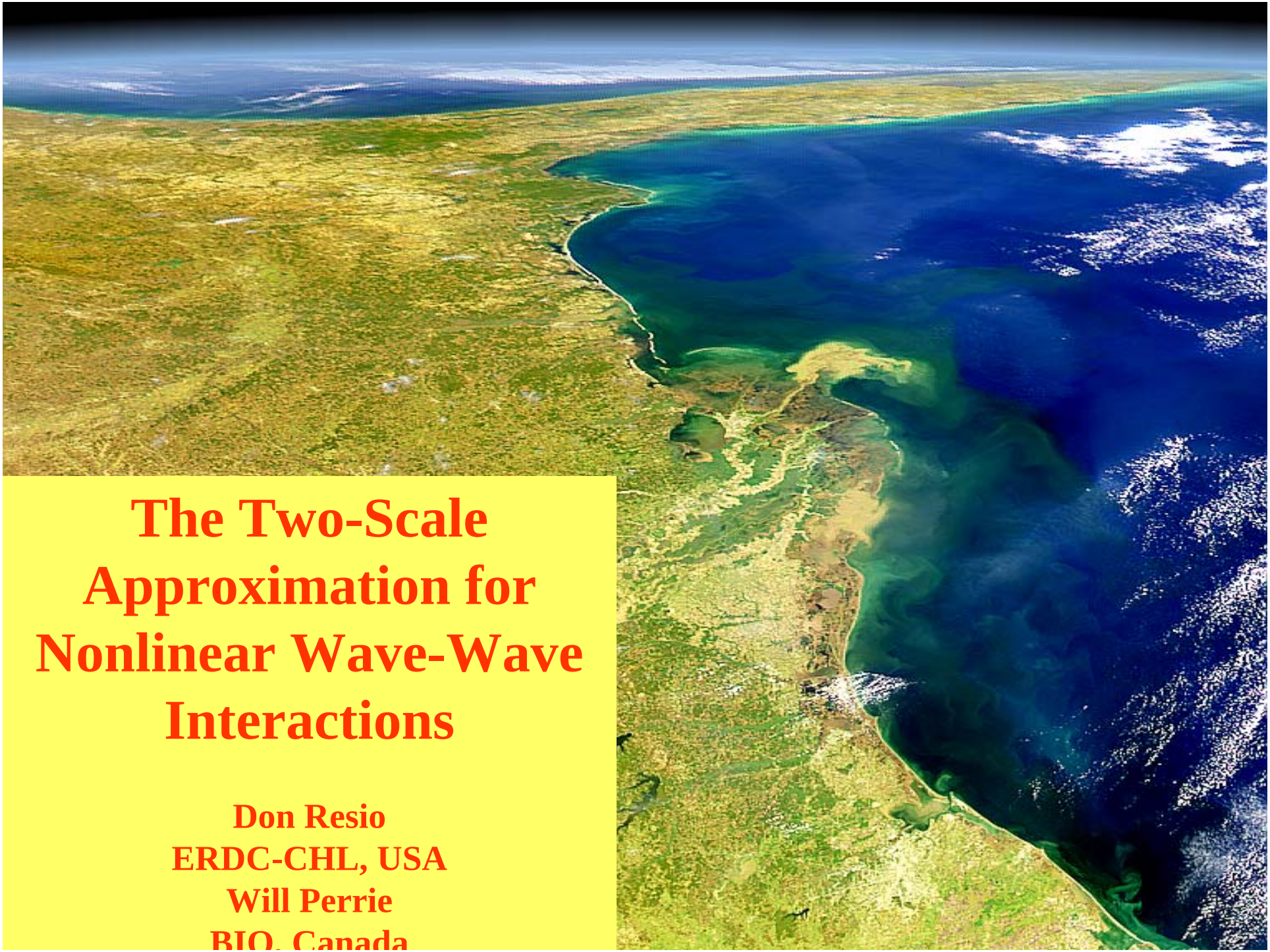


# The Two-Scale Approximation for Nonlinear Wave-Wave Interactions

**Don Resio**
**ERDC-CHL, USA**
**Will Perrie**
**BIO, Canada**

**Why did we switch from 2nd Generation spectral models to 3rd Generation spectral models?**

**To allow detailed balance calculations of source terms in order to be able to treat complex situations.**

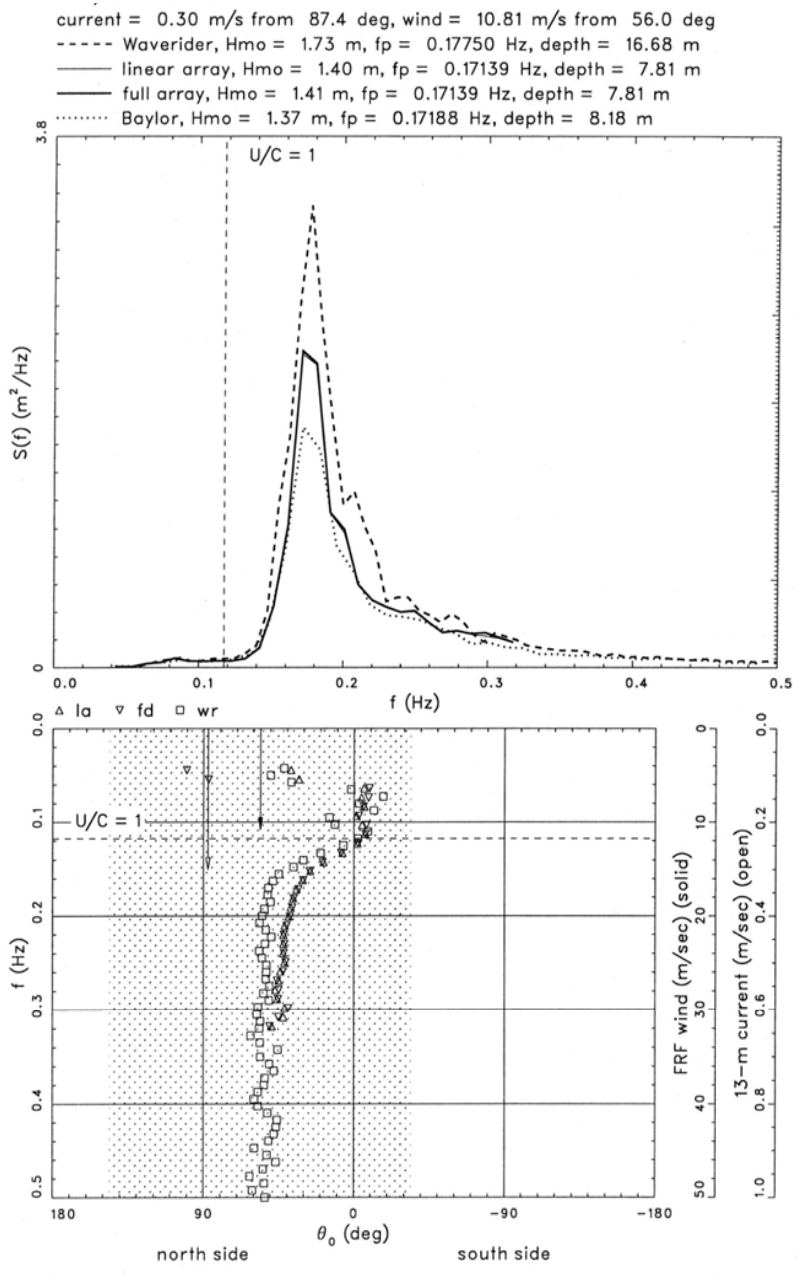
**The nonlinear interaction source term is critical to this balance.**

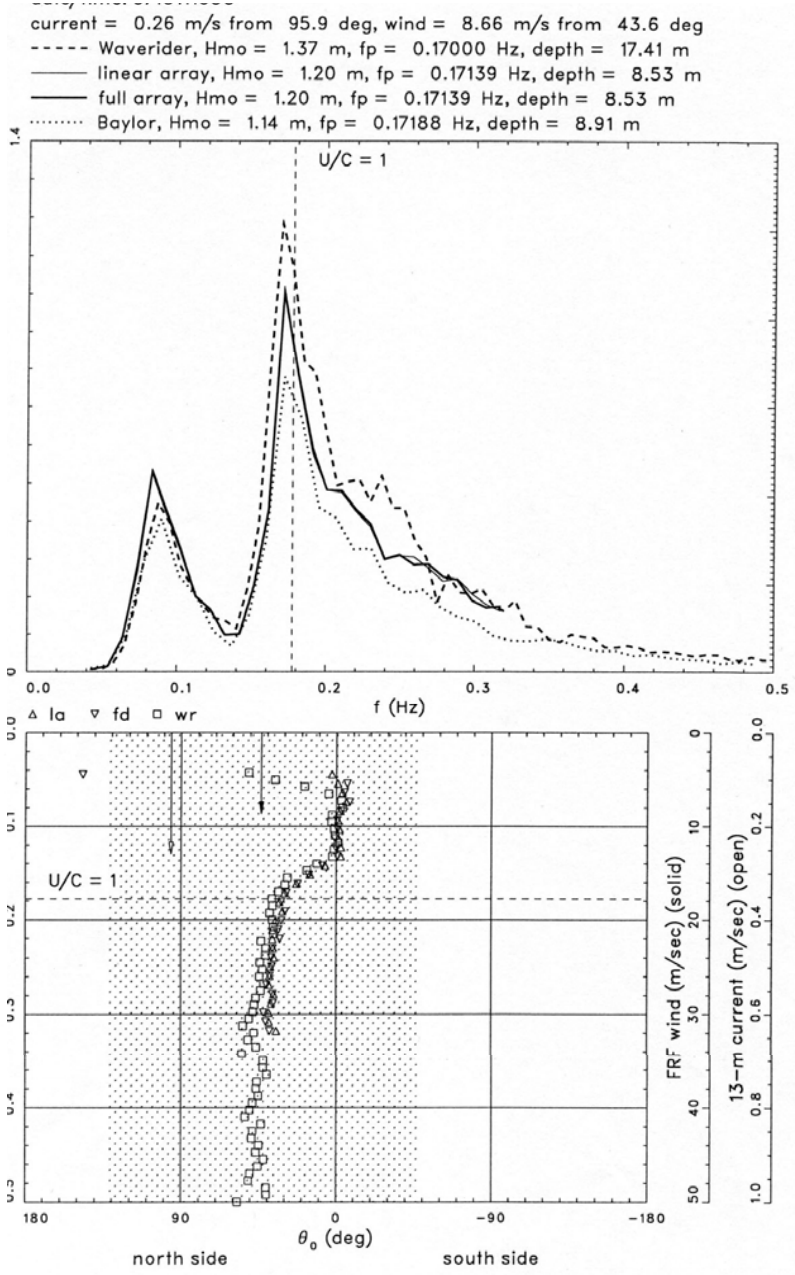
**If this term is incorrect, all the others must be tuned to compensate. This is not physics --- this is tuning.**

**<u>Objective of this paper</u>**

**Examine the accuracy of the DIA and its relationship to the full integral.**

**Introduce a new approximation method for nonlinear interaction source term with improved accuracy, while retaining "similar" computational efficiency**




Coastal wave spectra can be very complex and contain directional shear!
As can wave spectra in hurricanes/turning winds/slanting fetch/limited fetch

# Even “simple” spectra can have relatively complex shapes!

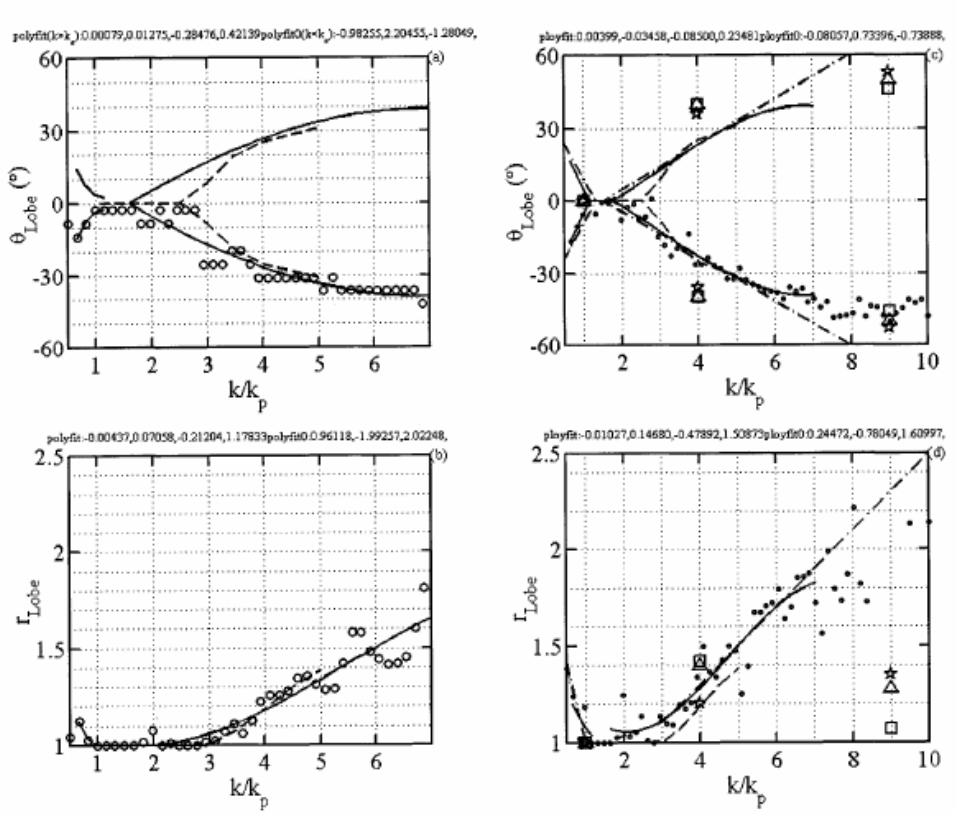
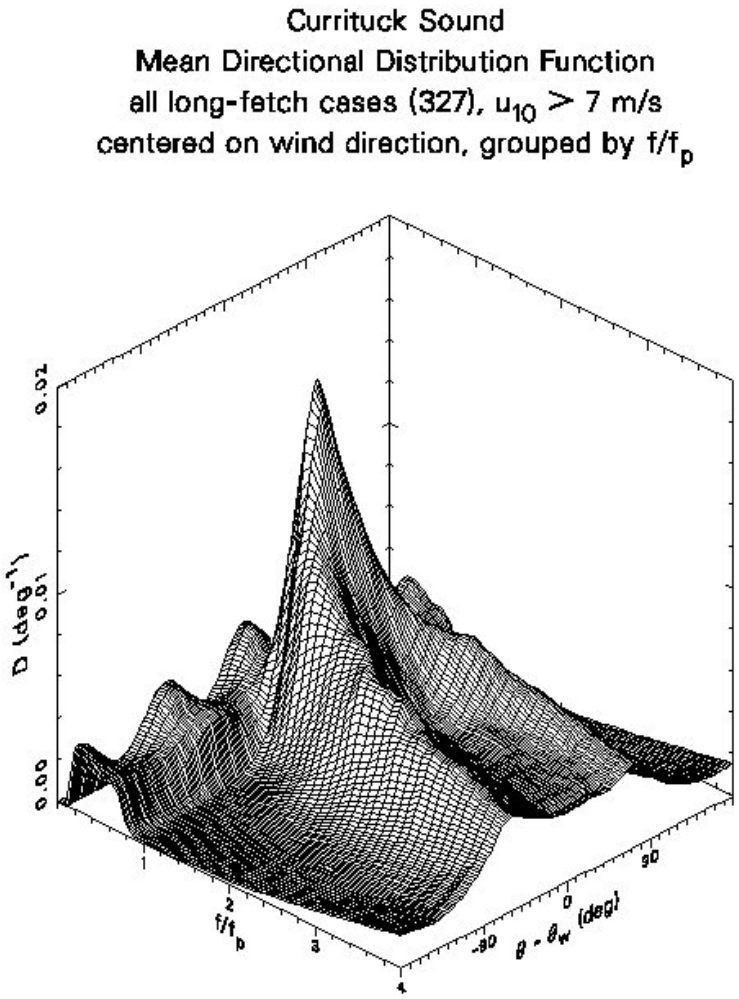
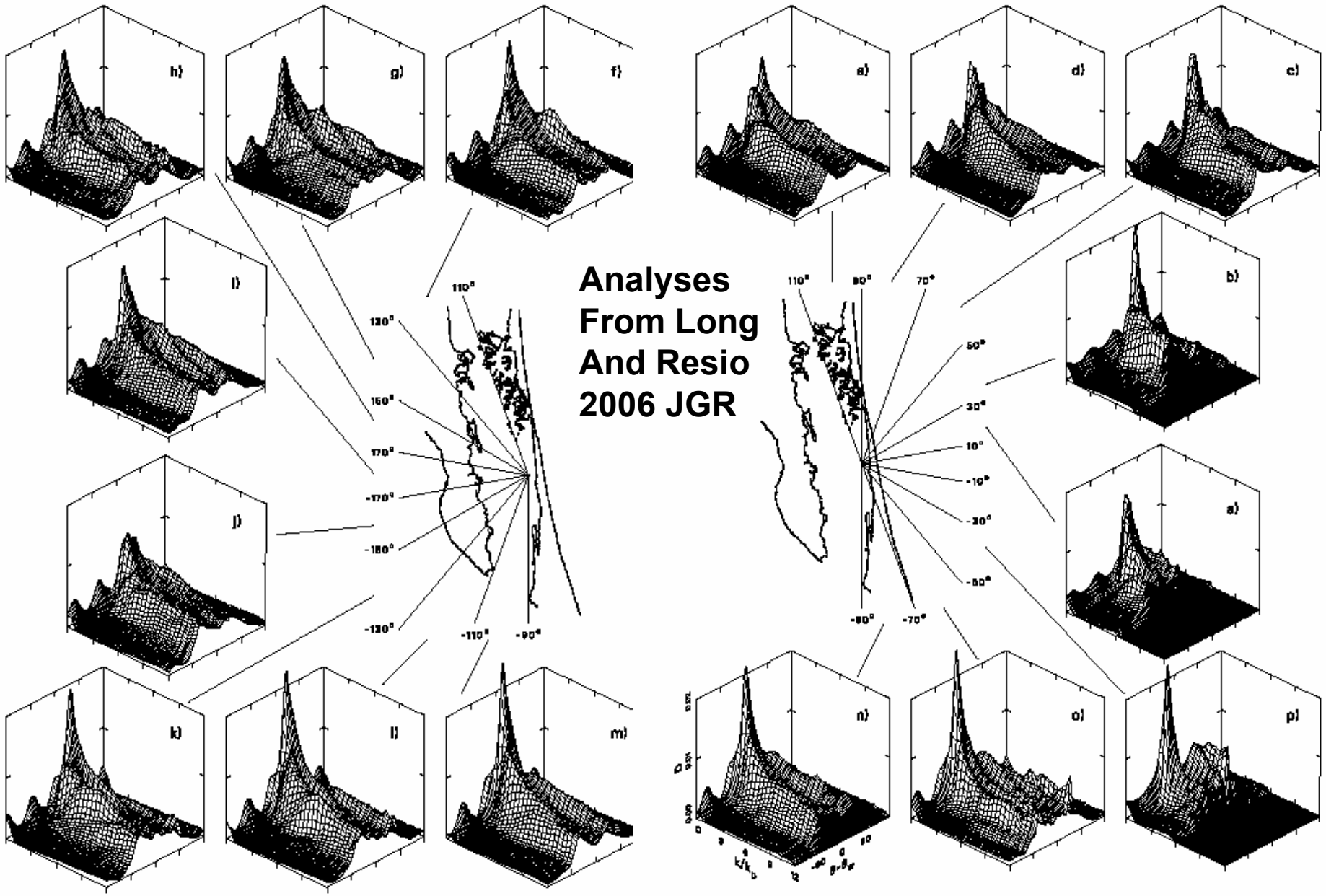
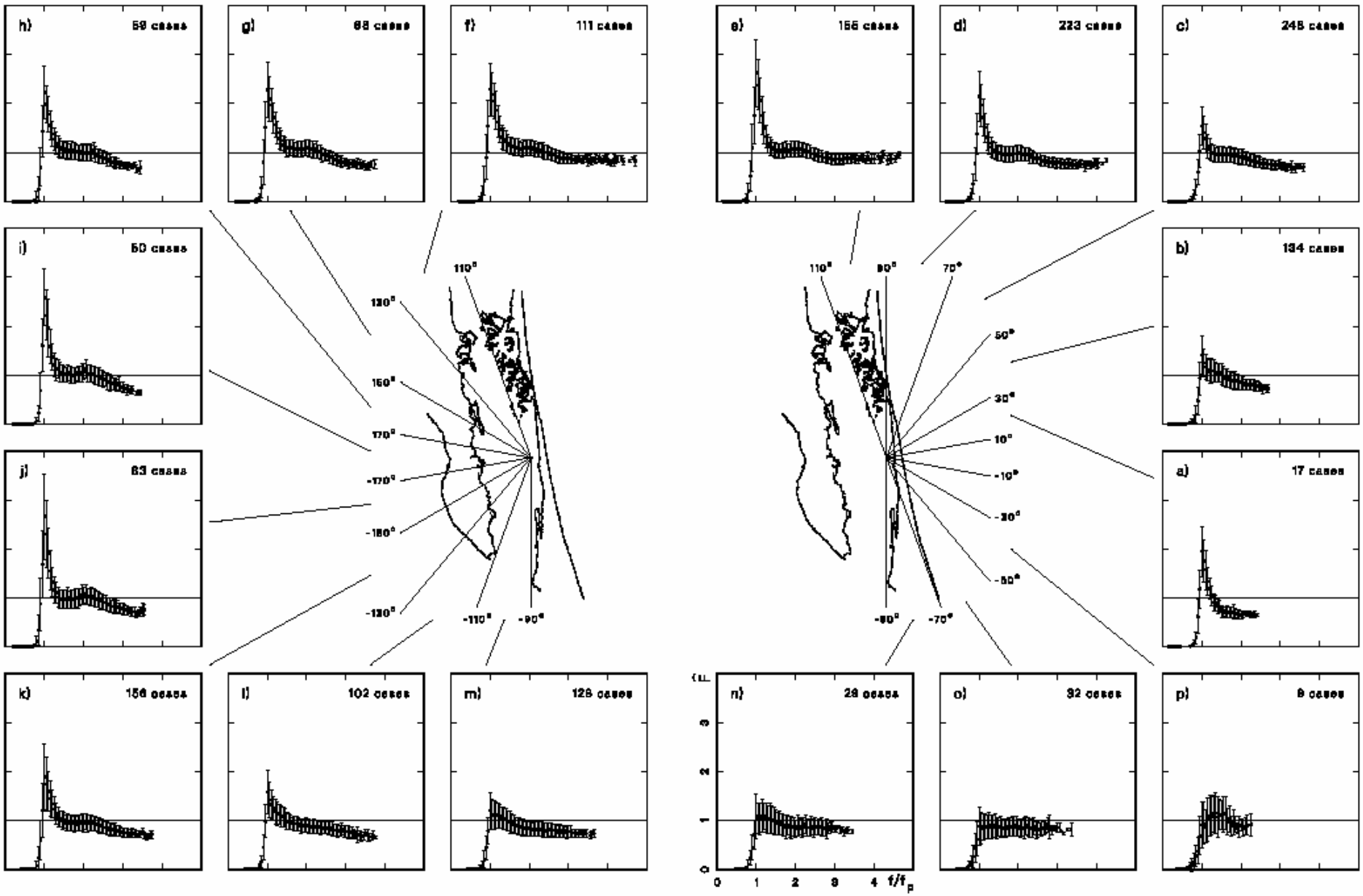



FIG. 10. The lobe angle (a, c) and the lobe ratio (b, d) of the bimodal distribution. In (a) and (b), the directional resolution is degraded to have a uniform resolution as done for the Fourier decomposition procedure. Solid curves: from polynomial fitting (coefficients listed in Table 2), and dashed curves: computed by $D_{EFFA}(\theta)$. In (c) and (d), the directional resolution is not degraded. The dashed-and-dotted curves are computed from Eqs. (20) and (21). Numerical results of Banner and Young (1994) on the effect of dissipation functions are shown with stars: quadratic, triangles: cubic, and square: quartic frequency dependence. Quasi-steady wave field.




Analyses
From Long
And Resio
2006 JGR


h) 59 cases
g) 88 cases
f) 111 cases
e) 156 cases
d) 223 cases
c) 245 cases
i) 50 cases
b) 134 cases
j) 83 cases
a) 17 cases
k) 156 cases
l) 102 cases
m) 128 cases
n) 28 cases
o) 92 cases
p) 9 cases
110°
130°
150°
170°
-170°
-150°
-130°
-110°
-90°
90°
70°
50°
30°
10°
-10°
-30°
-50°
-70°
f/f_p

***General 4-wave interaction integral for rate of change of action density at $k_1$***

Involves action densities at 4 wavenumbers, the coupling coefficient (C) which is also a function of the wavenumbers, and a Jacobian tranform for moving from k to (s,n) space

$$\frac{\partial n(\underline{k}_1)}{\partial t} = \iint T(\underline{k}_1, \underline{k}_3) d\underline{k}_3$$

$$T(\underline{k}_1, \underline{k}_3) = \oint [n_1 n_3 (n_4 - n_2) + n_2 n_4 (n_3 - n_1)] C(\underline{k}_1, \underline{k}_2, \underline{k}_3, \underline{k}_4) \left| \frac{\partial W}{\partial \underline{n}} \right|^{-1} d\underline{s}$$

The integral for 4-wave interactions can be written as an integral of the transfers from one wavenumber into another, where 2 other wave numbers are free to assumed continuous values along loci determined by the resonant conditions:

$$\underline{k}_4 = \underline{k}_1 + \underline{k}_2 - \underline{k}_3$$

$$\underline{k}_2 = \underline{k}_2 (s, \underline{k}_1, \underline{k}_3) \longrightarrow \text{DIA Constraint}$$

$$\vec{k}_1 - \vec{k}_3 = \vec{k}_4 - \vec{k}_2 \quad as \quad \vec{k}_2 \to \vec{k}_1$$

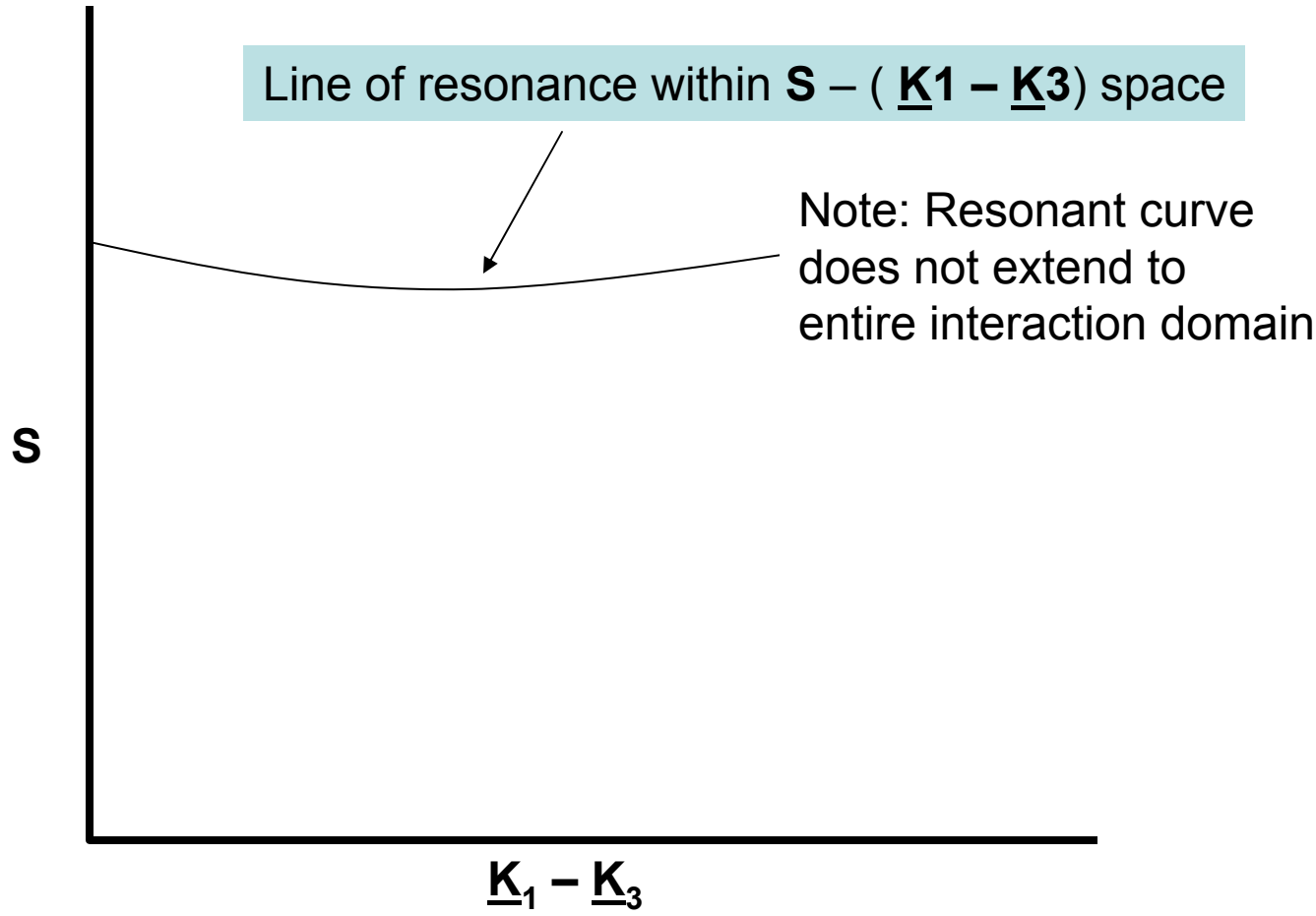
Where s is determined from W=0 solution

$$W = \omega_1 + \omega_2 - \omega_3 - \omega_4$$

***DIA***

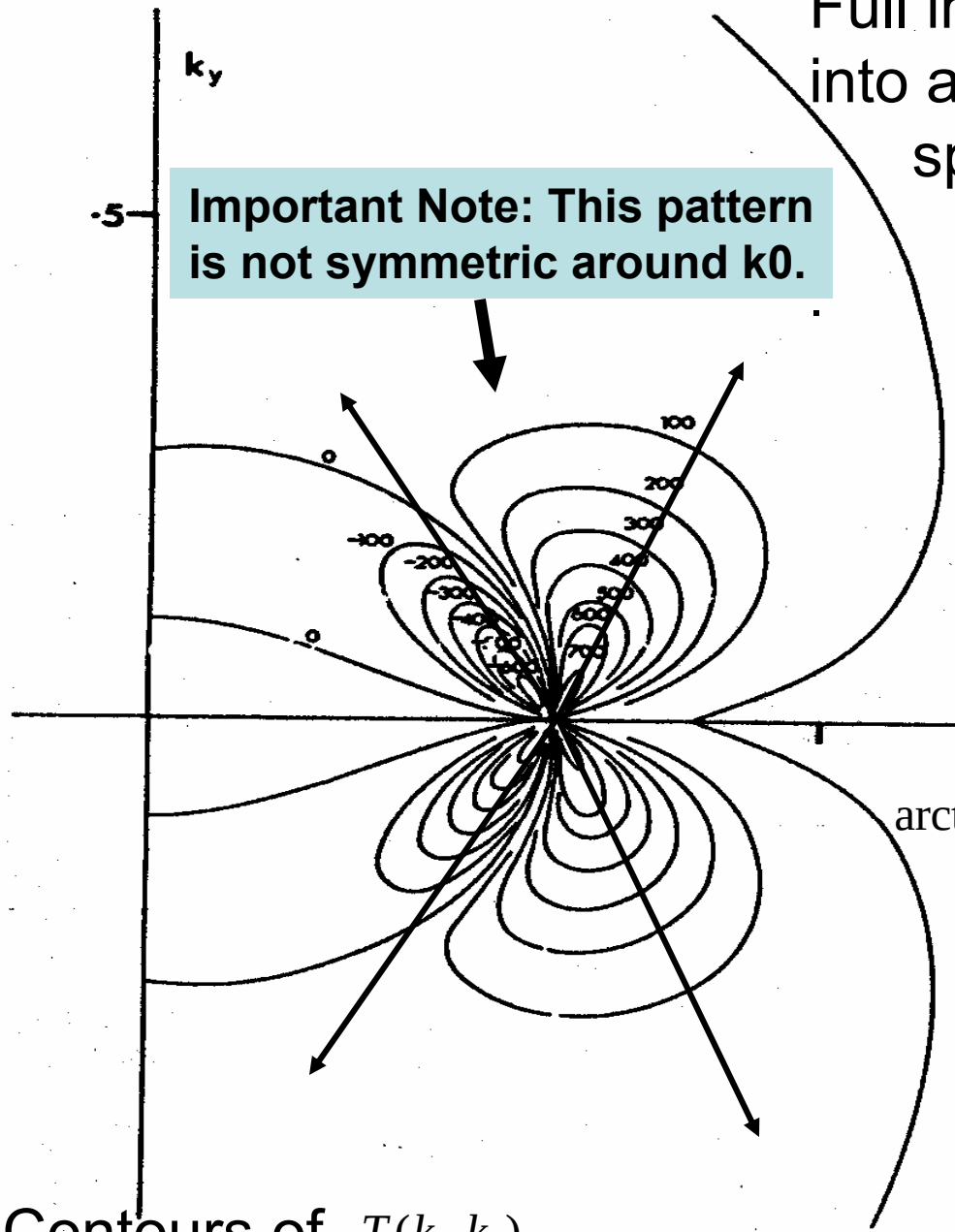
Represents subset of points for which two wavenumbers are (close to) equal. Essentially gives the interactions along Phillips' "figure-8" curve.

**How are these two sets of interactions related?**




**There is one point along each s-locus for which k2=k1, up to the limit of the maximum k3 value in the Figure 8 resonance pattern. The Full Integral considers contributions over the 2-dimensions above. The DIA only samples along the line.**

# Full integral estimate for tranfers into and out of a location near the spectral peak (Webb, 1978)




Transfers are very comparable to the results of Longuet-Higgins and Fox (1975) based on Davey and Stewartson (1974) work on a 3-dimensional wave packet instability. Note asymptotic angles of transfer maxima at

$$\arctan\left(\frac{1}{\sqrt{2}}\right) = \text{asymptote of "Figure 8" pattern}$$

Contours of $T(\underline{k}_1, \underline{k}_3)$

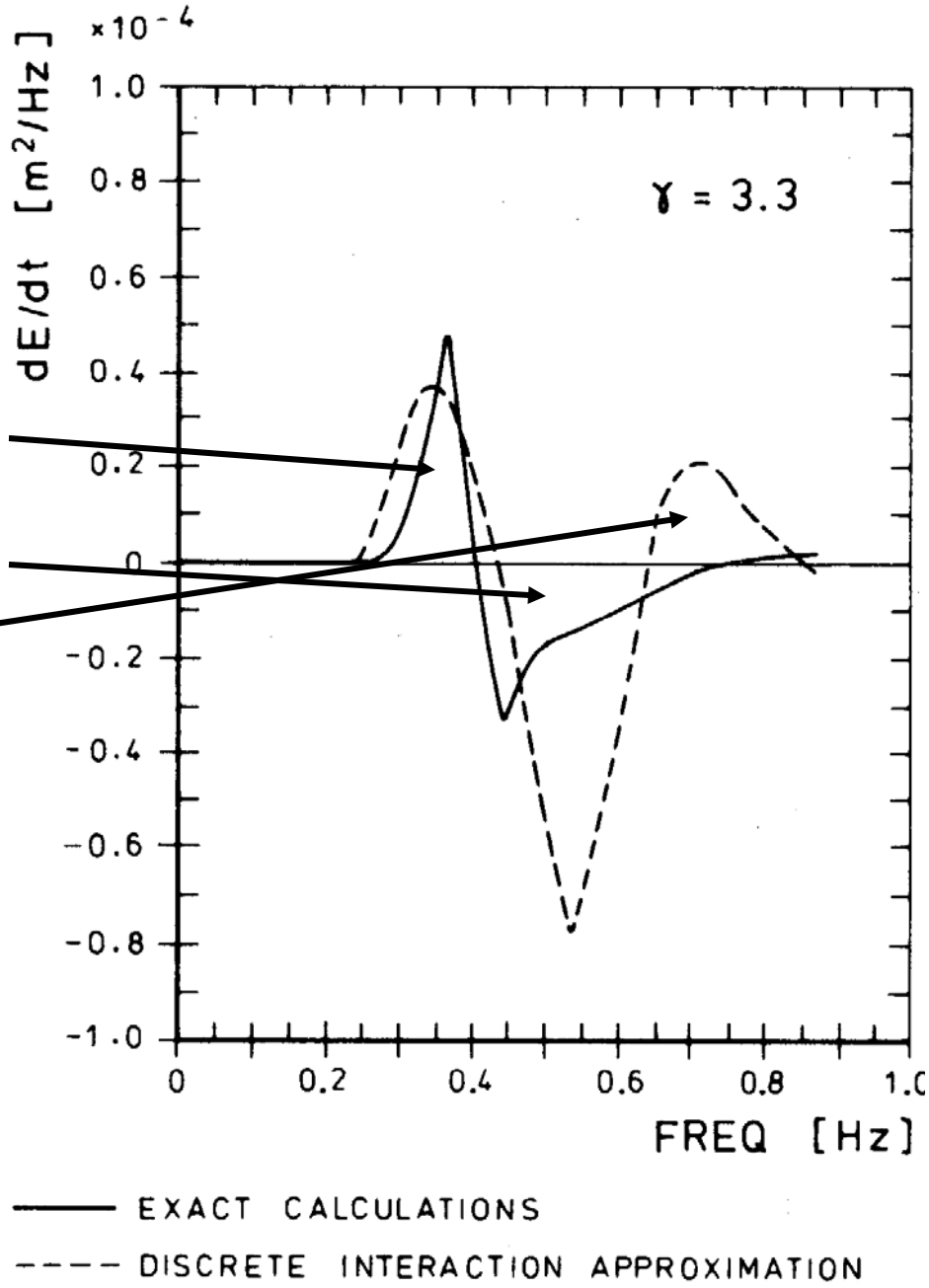
**DIA is parametrically calibrated to match only a single quantity – the total energy transferred to the forward face of the spectrum.**

**<u>but:</u>**

Calibrated to “match” here

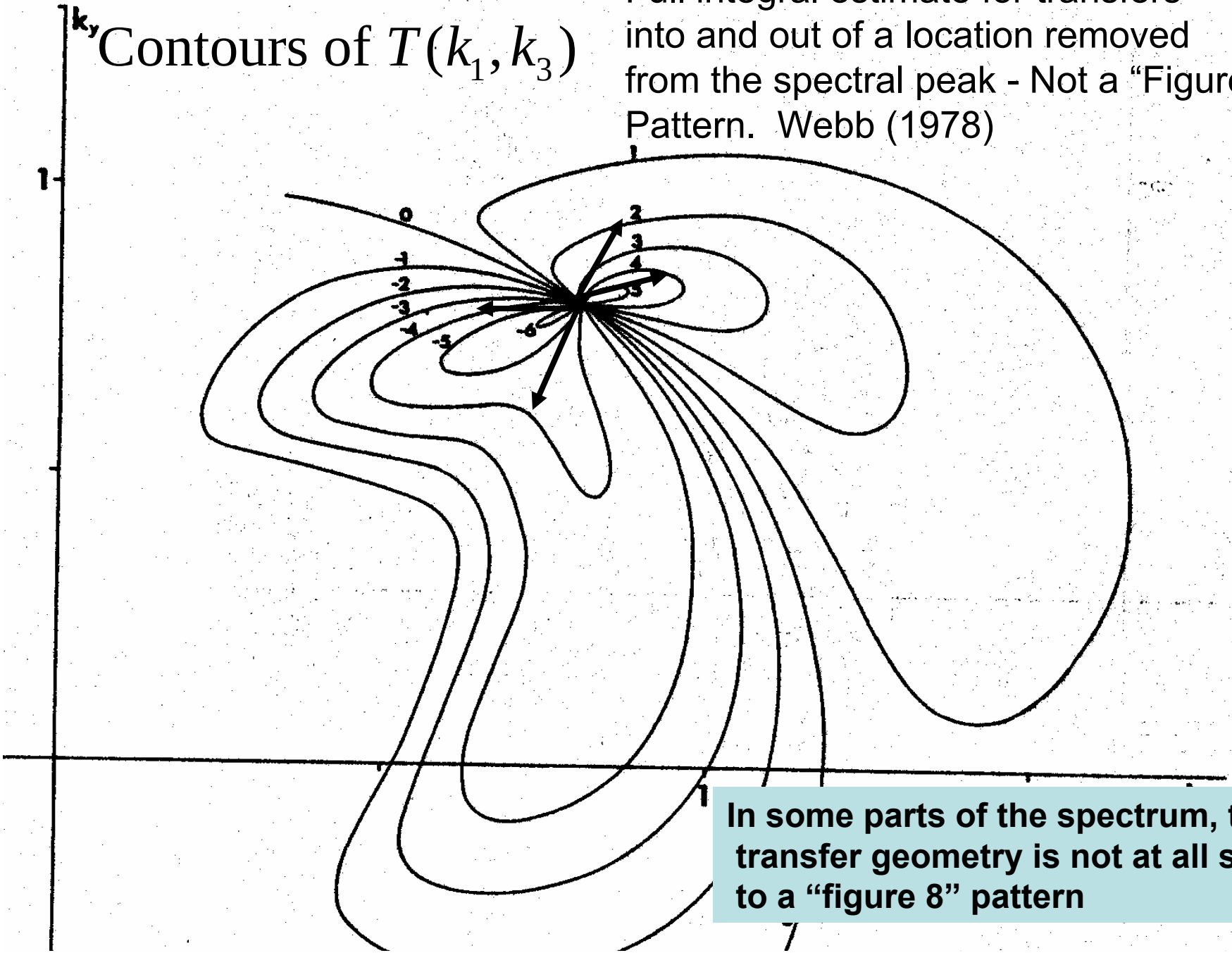
Creates a mis-match here

And here



# Contours of $T(k_1, k_3)$

Full integral estimate for transfers into and out of a location removed from the spectral peak - Not a “Figure 8” Pattern. Webb (1978)

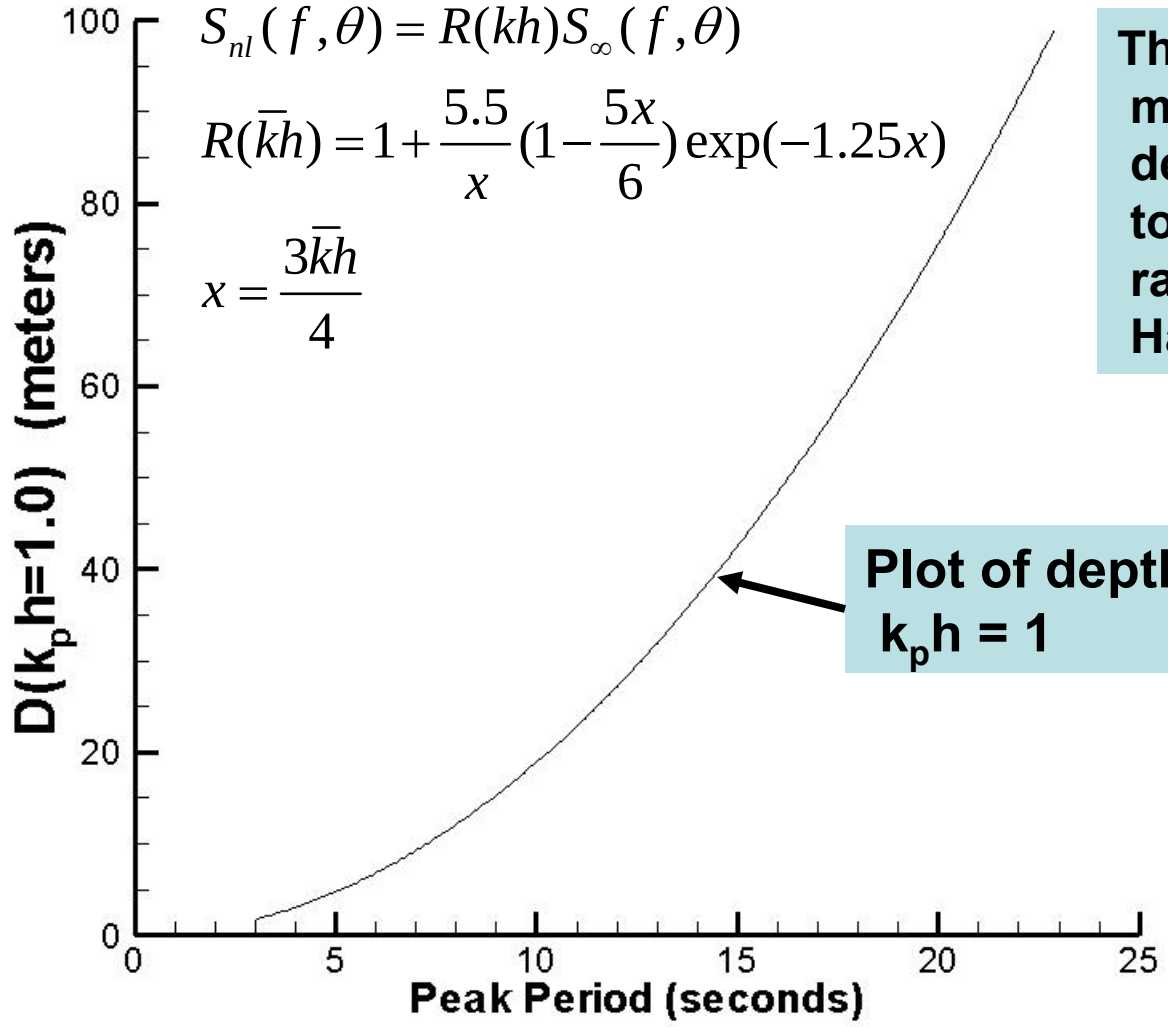



**In some parts of the spectrum, the transfer geometry is not at all similar to a “figure 8” pattern**

$$S_{nl}(f,\theta) = R(\bar{k}h)S_{\infty}(f,\theta)$$

$$R(\bar{k}h) = 1 + \frac{5.5}{x}\left(1 - \frac{5x}{6}\right)\exp(-1.25x)$$

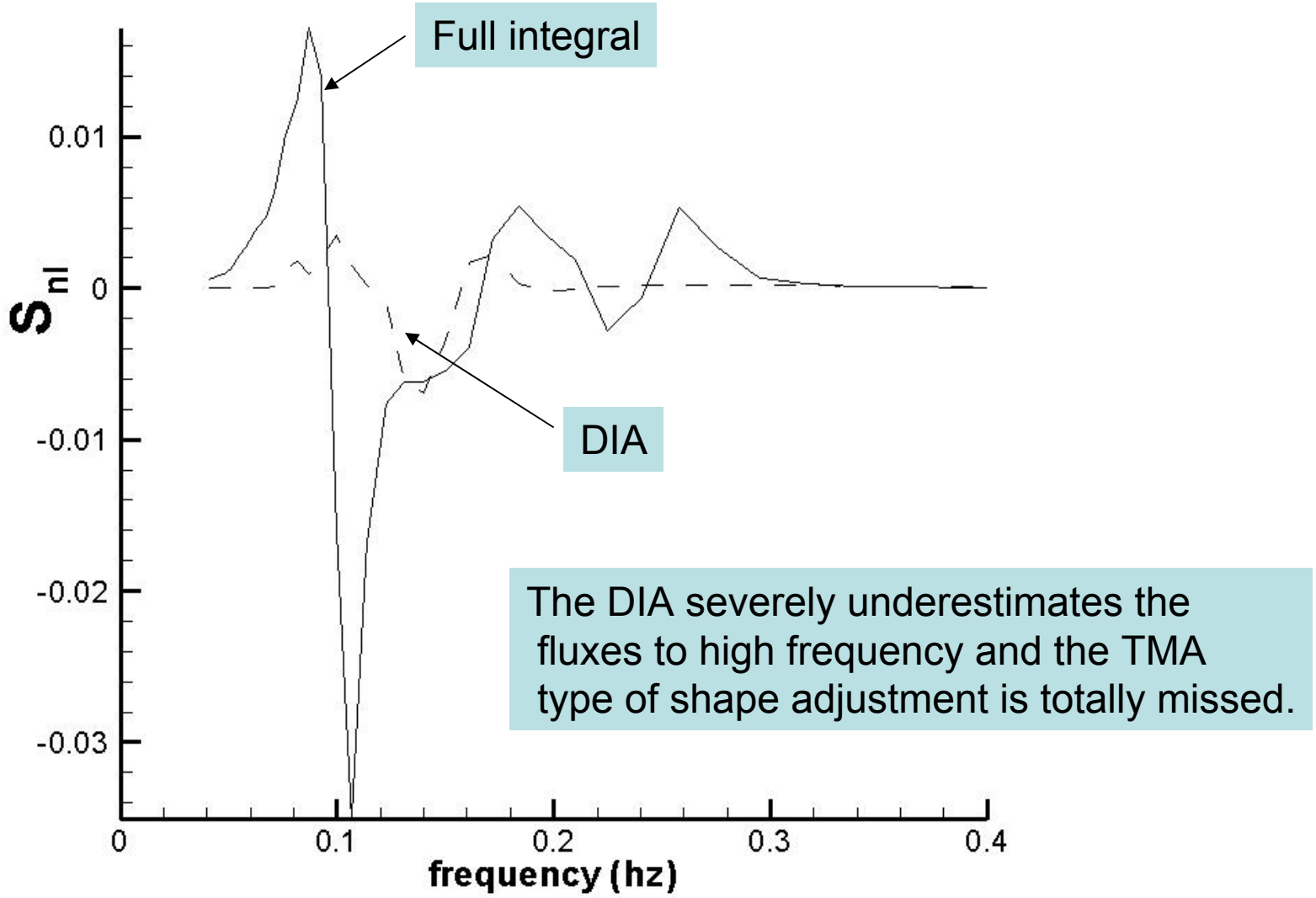
$$x = \frac{3\bar{k}h}{4}$$




**The DIA uses a parametric methods to adjust the deep water transfer rates to shallow-water transfer rates – Herterich & Hasselmann (1980).**

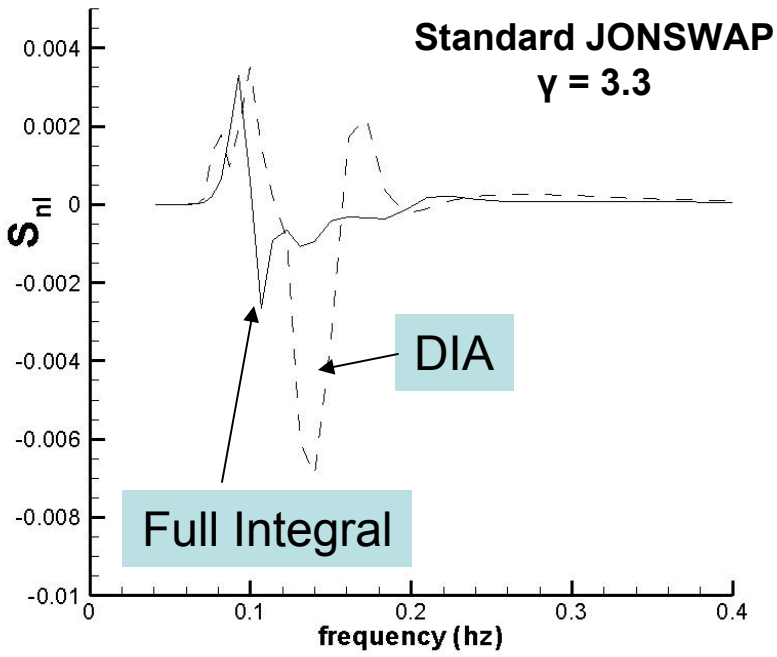
**Plot of depth at which $k_p h = 1$**

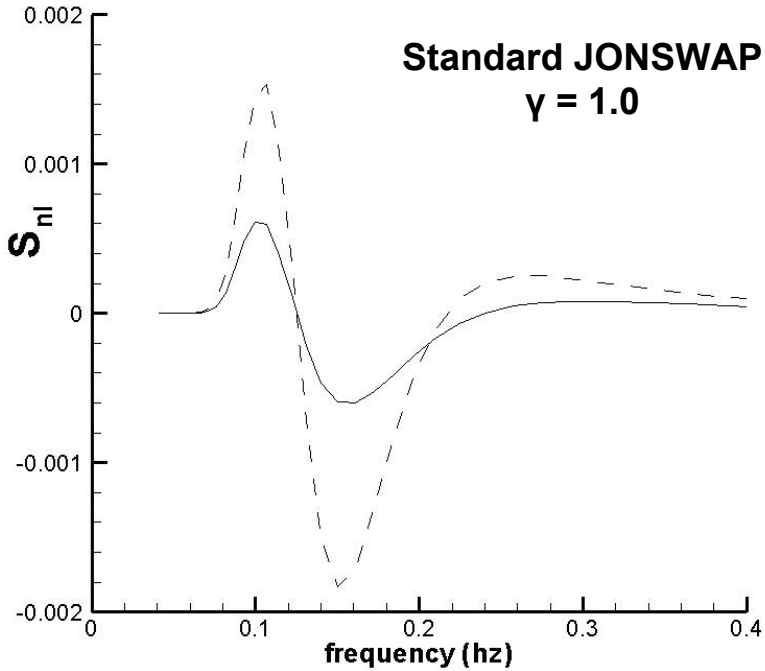
# What happens if you violate this limit?

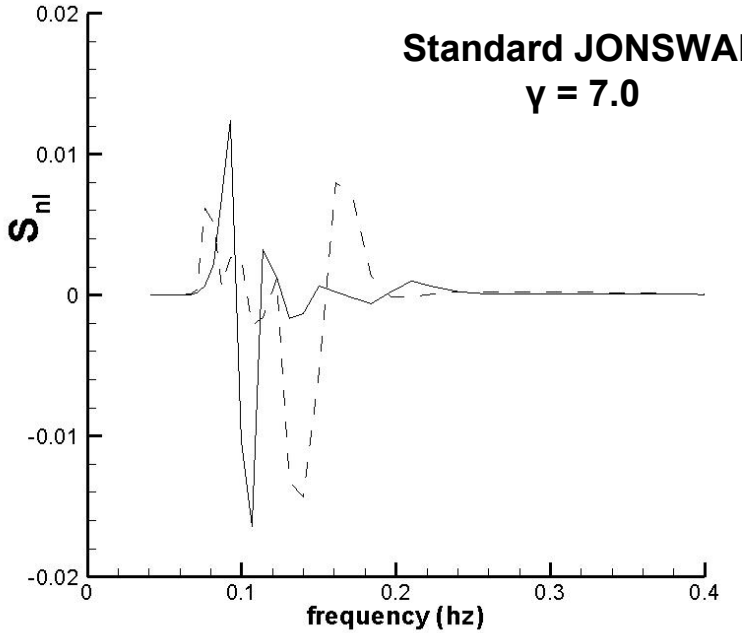



Comparison of a DIA calculation and an actual full-integral, finite-depth calculation for the case of kph=0.7 (JONSWAP spectrum with a peak period of 10 seconds in depth of 10.5 meters).

***Since the DIA only samples a slice from the complete integral, and is tuned to fit a specific spectrum, how well does it work for a range of peakedness typical of wave generation conditions?***

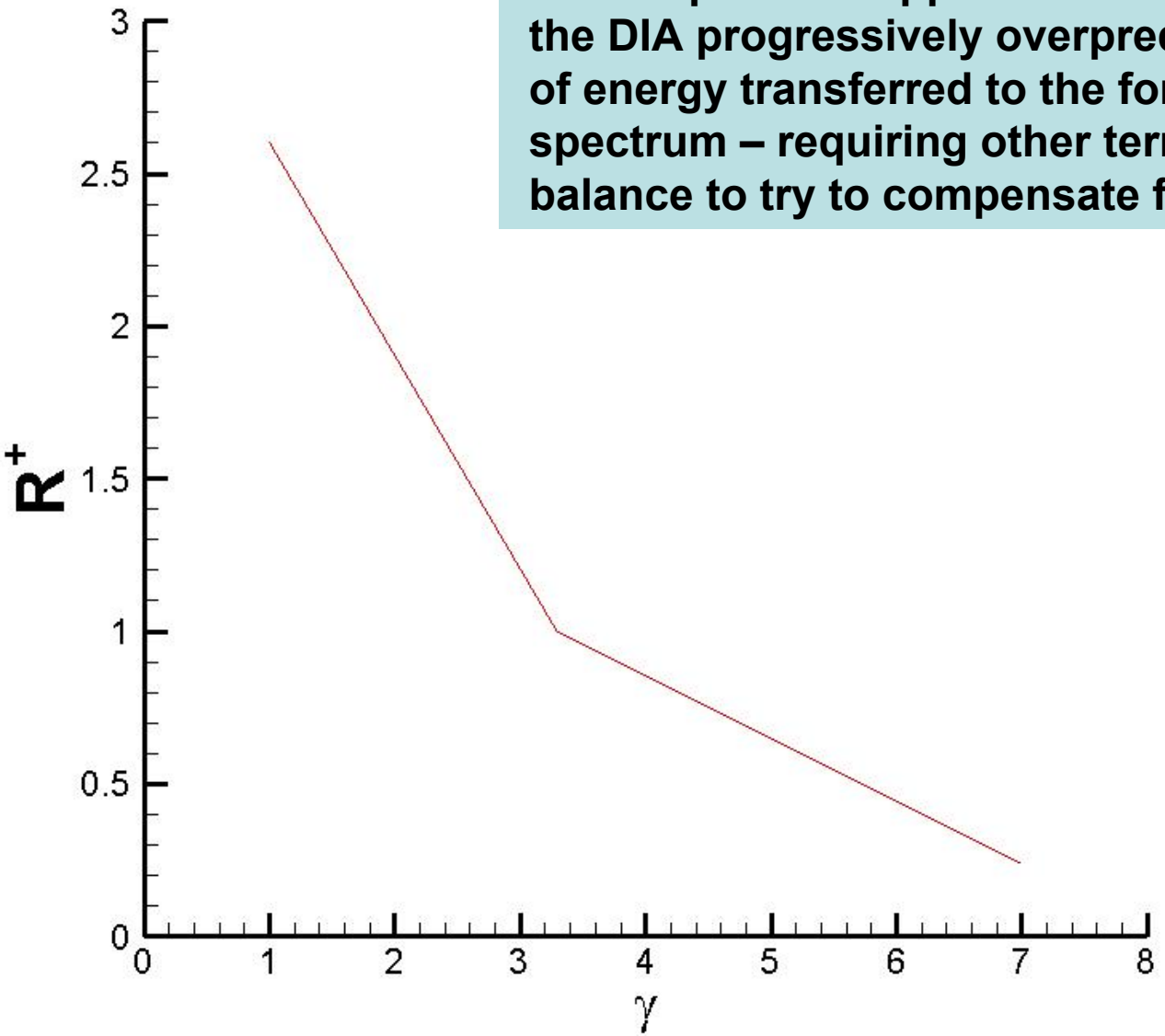


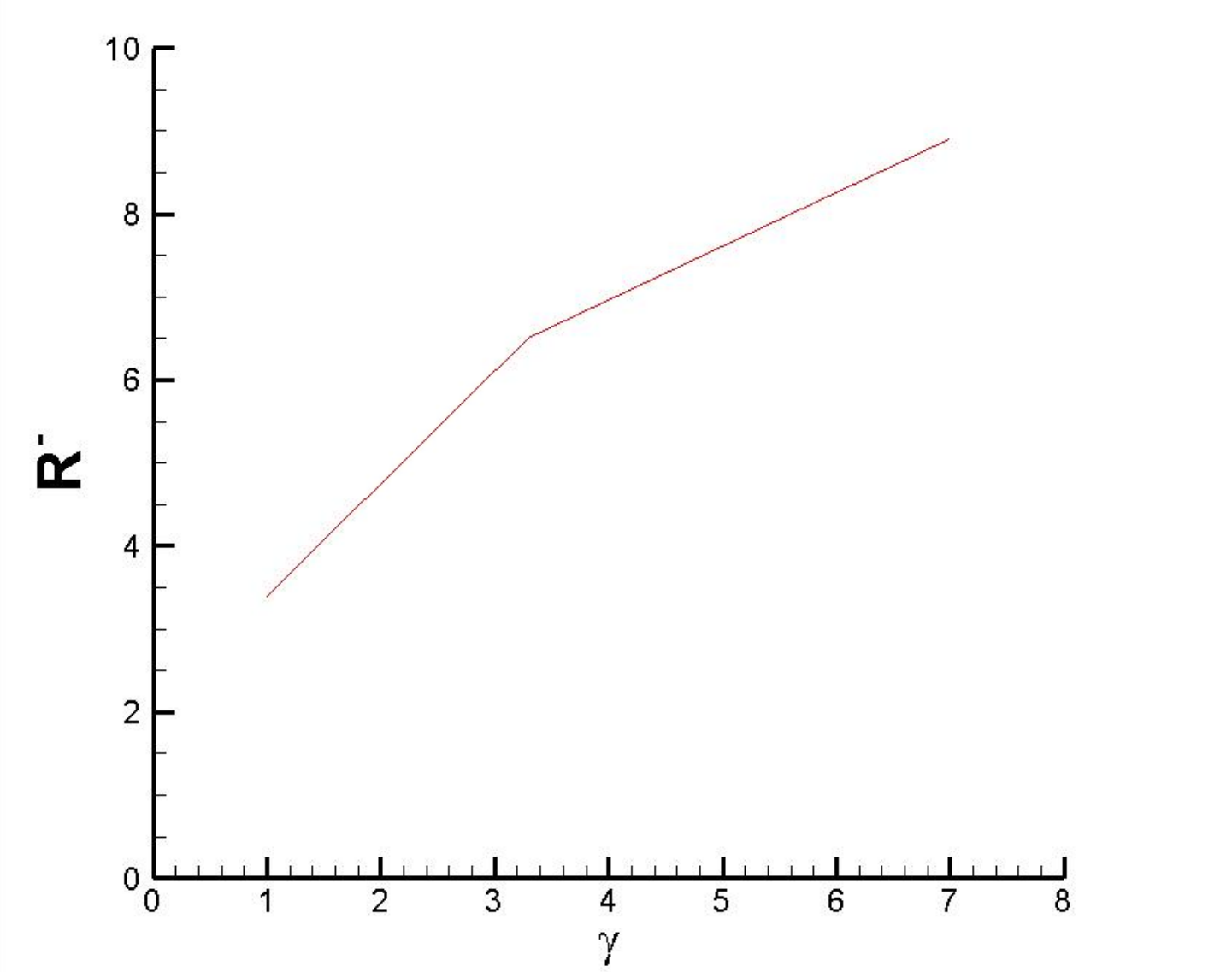




**DIA performance for JONSWAP spectra with selected peakedness values. Solid line is full integral. Dashed line is DIA. This does not provide a consistent amount of energy to forward face of spectrum.**

**As the spectrum approaches full development the DIA progressively overpredicts the amount of energy transferred to the forward face of the spectrum – requiring other terms in the detailed balance to try to compensate for this transfer.**




The ratio of the maximum value within the positive lobe predicted by the DIA to the maximum value predicted by the full integral, as a function of JONSWAP spectral peakedness parameter, γ.




Same as previous figure, defining a ratio for the largest negative values within the negative lobe, as a function of JONSWAP spectral peakedness parameter, γ.

**Problem with DIA – The basis is not the integral that we are trying to estimate.**

**We need a new approximation that;**

- **conserves constants of motion (action, energy, momentum)**
- **has its basis in the correct integral (important for complex cases)**
- **retains the number of degrees of freedom in the modeled spectrum**
- **is not limited to $k_p h \geq 1$**
- **is much more efficient than the full integral**

# Basis for Two-Scale Approximation

$$
\begin{aligned}
N^3 = {} & \hat{n}_1\hat{n}_3(\hat{n}_4-\hat{n}_2)+\hat{n}_2\hat{n}_4(\hat{n}_3-\hat{n}_1)+ \\
& n_1'n_3'(n_4'-n_2')+n_2'n_4'(n_3'-n_1')+ \\
& \hat{n}_1\hat{n}_3(n_4'-n_2')+\hat{n}_2\hat{n}_4(n_3'-n_1')+ \\
& n_1'n_3'(\hat{n}_4-\hat{n}_2)+n_2'n_4'(\hat{n}_3-\hat{n}_1)+ \\
& \hat{n}_1n_3'(\hat{n}_4-\hat{n}_2)+\hat{n}_2n_4'(\hat{n}_3-\hat{n}_1)+ \\
& n_1'\hat{n}_3(\hat{n}_4-\hat{n}_2)+n_2'\hat{n}_4(\hat{n}_3-\hat{n}_1)+ \\
& \hat{n}_1n_3'(n_4'-n_2')+\hat{n}_2n_4'(n_3'-n_1')+ \\
& n_1'\hat{n}_3(n_4'-n_2')+n_2'\hat{n}_4(n_3'-n_1')
\end{aligned}
$$

$$n = \hat{n} + n'$$

$$S_{nl}(f,\theta) = B + L + X$$

**Line 1 contains interactions for only B**
**Line 2 contains interactions for only L**
**Lines 3-8 contain cross-interactions between B and L**

**This approximation to the full integral would be exact if all terms were retained.**

$$\frac{\partial n_1}{\partial t} = \iint \oint N_A^3 C \left| \frac{\partial W}{\partial n} \right|^{-1} ds \; k_3 d\theta_3 dk_3$$

Scale 1 Terms

$$+$$

$$\iint \oint N_B^3 C \left| \frac{\partial W}{\partial n} \right|^{-1} ds \; k_3 d\theta_3 dk_3$$

Retained Terms from L+X

$$+$$

$$\iint \oint N_C^3 C \left| \frac{\partial W}{\partial n} \right|^{-1} ds \; k_3 d\theta_3 dk_3$$

Neglected Terms from L+X

The fundamental idea here is to capture the broad-scale distribution of energy parametrically and to allow “local” differences to be treated as shown below. Terms that are neglected tend to contribute in a +/- sense around s.

This could be a DIA form or a diffusion operator, but we would lose considerable accuracy.

neglected $n_2'$ and $n_4'$ terms

$$\frac{\partial n_1}{\partial t} = B + \iint N_*^3 C \left| \frac{\partial W}{\partial n} \right|^{-1} ds \; k_3 d\theta_3 dk_3 + \ldots\ldots$$

$N_*^3$ terms neglect terms containing $n_2'$ and $n_4'$ - retain $\hat{n}_2$ and $\hat{n}_4$

$$N_*^3 = \hat{n}_2 \hat{n}_4 (n_3' - n_1') + n_1' n_3' (\hat{n}_4 - \hat{n}_2) + \hat{n}_1 n_3' (\hat{n}_4 - \hat{n}_2) + n_1' \hat{n}_3 (\hat{n}_4 - \hat{n}_2)$$

Note: X is typically 2-3 times larger than L or B. This is why linear sums (neural networks, EOF’s, etc.) do not work well for SnI estimation.

$$\frac{\partial n_1}{\partial t} = B + \left(\frac{k}{k_0}\right)^{19/2} \left\langle \left(\frac{\beta}{\beta_0}\right) \iint (\hat{n}_1 n_3' + n_1' \hat{n}_3 + n_1' n_3') \Lambda(\hat{n}_2 - \hat{n}_4, \underline{k}_1, k_*, \theta_*, \mathrm{x}_1, ..., \mathrm{x}_n)\, k_* d\theta_* dk_* \right.$$
$$\left. + \left(\frac{\beta}{\beta_0}\right)^2 \iint (n_1' - n_3') \Lambda(\hat{n}_2 \hat{n}_4, \underline{k}_1, k_*, \theta_*, \mathrm{x}_1, ..., \mathrm{x}_n)\, k_* d\theta_* dk_* \right\rangle$$

where

$$\Lambda(\hat{n}_2 - \hat{n}_4, \underline{k}_1, k_*, \theta_*, \mathrm{x}_1, ..., \mathrm{x}_n) = \oint C \left| \frac{\partial W}{\partial n} \right|^{-1} (\hat{n}_4 - \hat{n}_2) ds$$

$$\Lambda(\hat{n}_2 \hat{n}_4, \underline{k}_1, k_*, \theta_*, \mathrm{x}_1, ..., \mathrm{x}_n) = \oint C \left| \frac{\partial W}{\partial n} \right|^{-1} \hat{n}_2 \hat{n}_4 ds$$

$$\frac{\partial n_1}{\partial t} = \sum_{\delta f_i} \sum_{\delta \theta_j} \mu_{ij\vec{s}p} N_{*_P}^3 + \mu_{ij\vec{s}d} N_{*_D}^3$$

Pumping Diffusion

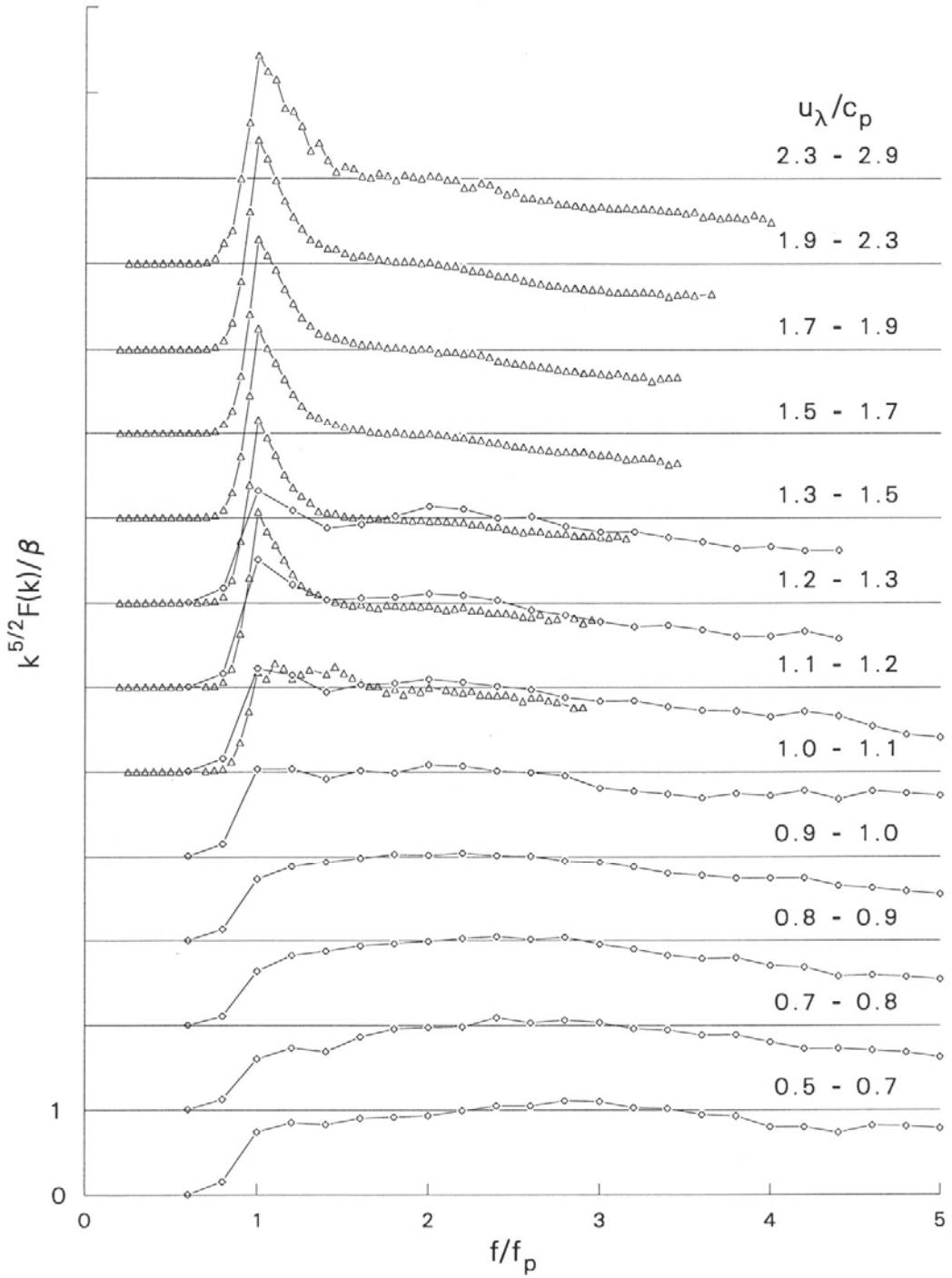
Note: This has no free parameters. All coefficients are determined from the full integral. Still conserves action, energy, and momentum since each elemental transfer is conservative.

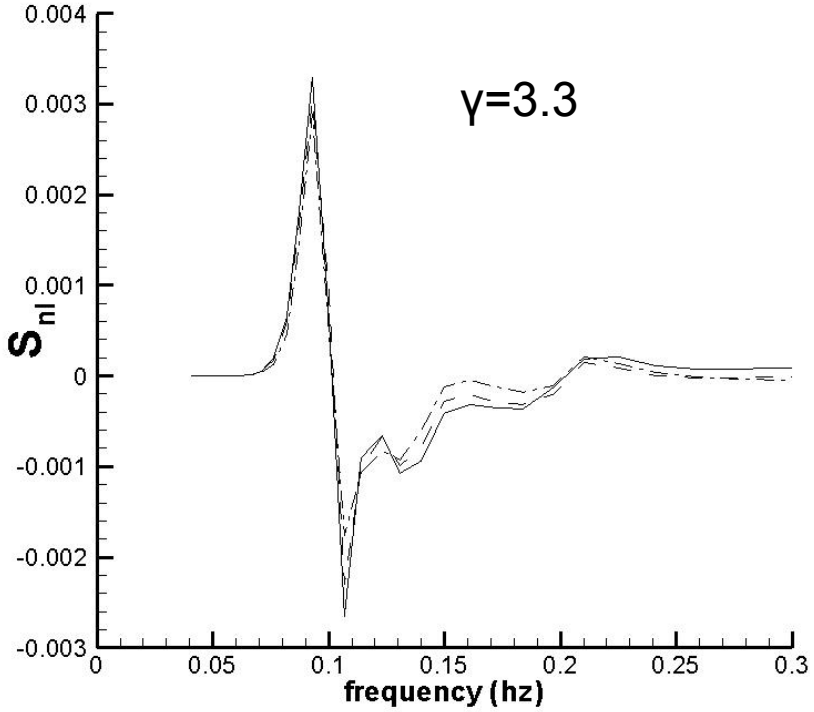
Note: computer time is about 3x that of DIA

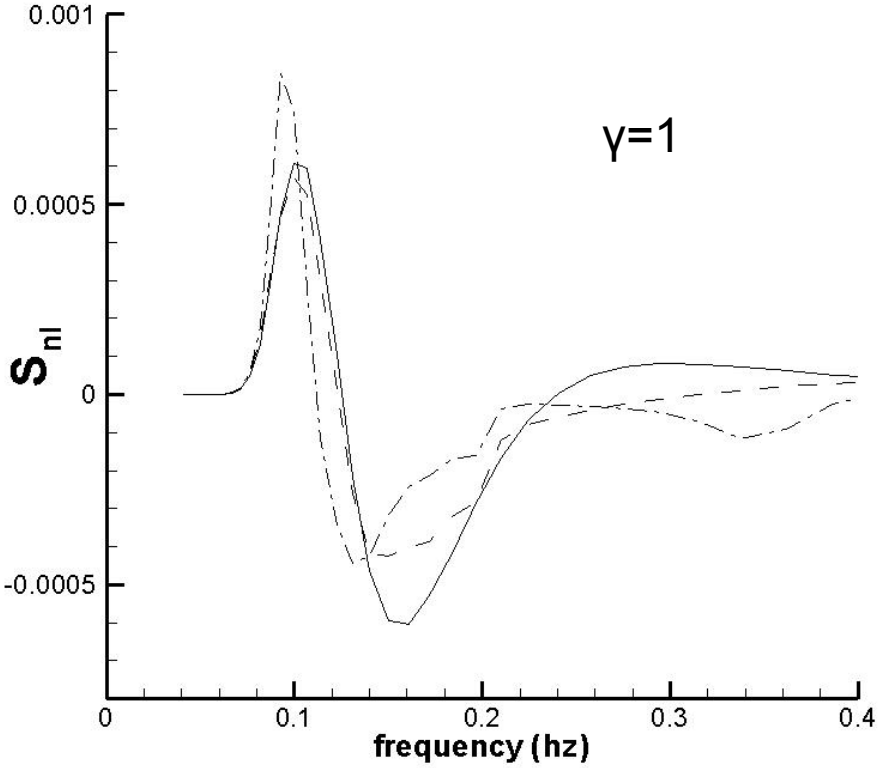
Approximation used here is a simple parameterization of B based on:

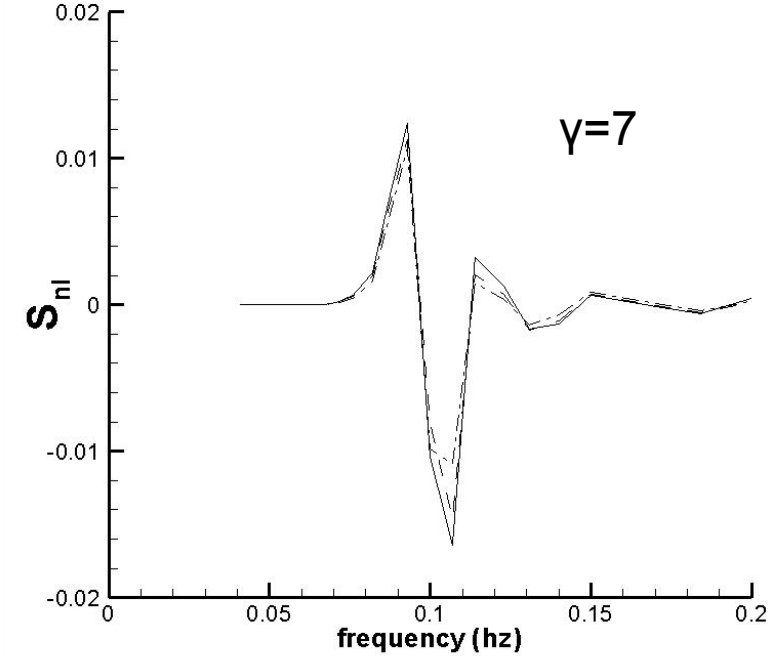
1 *dof* only (Peakedness)

Normalized spectra stratified by inverse wave age based on data from
Currituck Sound
Bering Sea
Atlantic Ocean

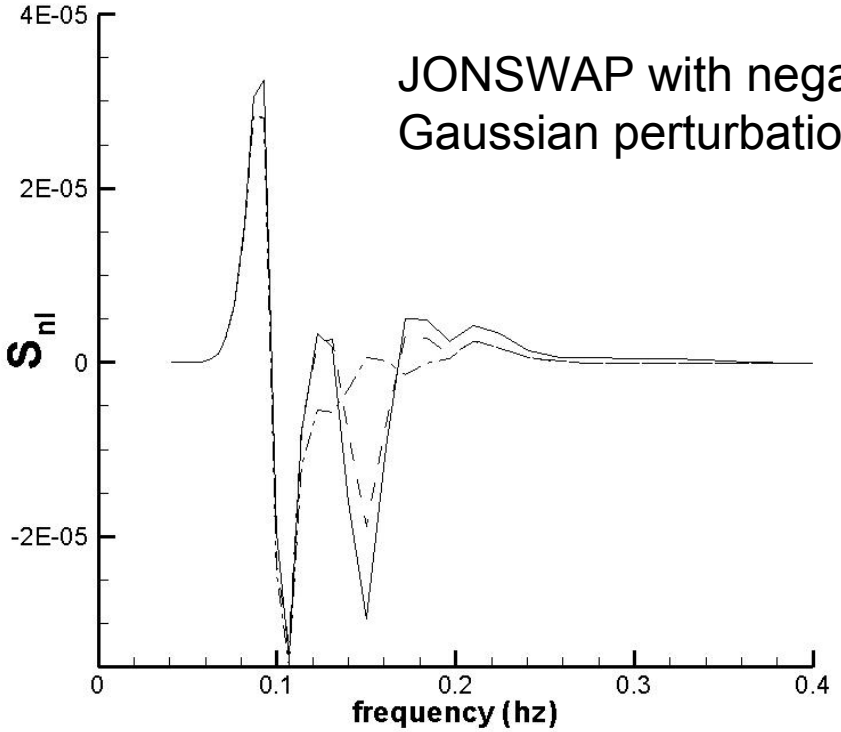




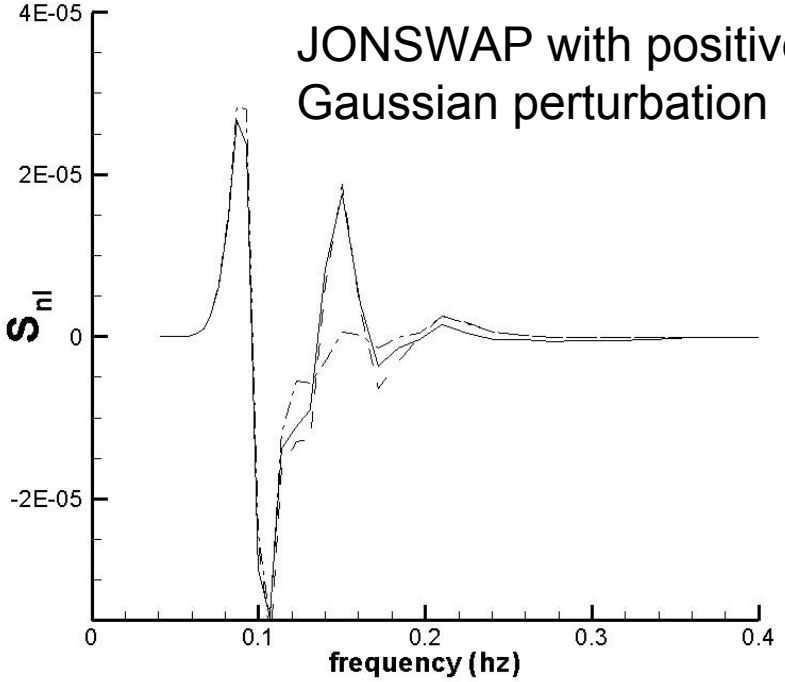



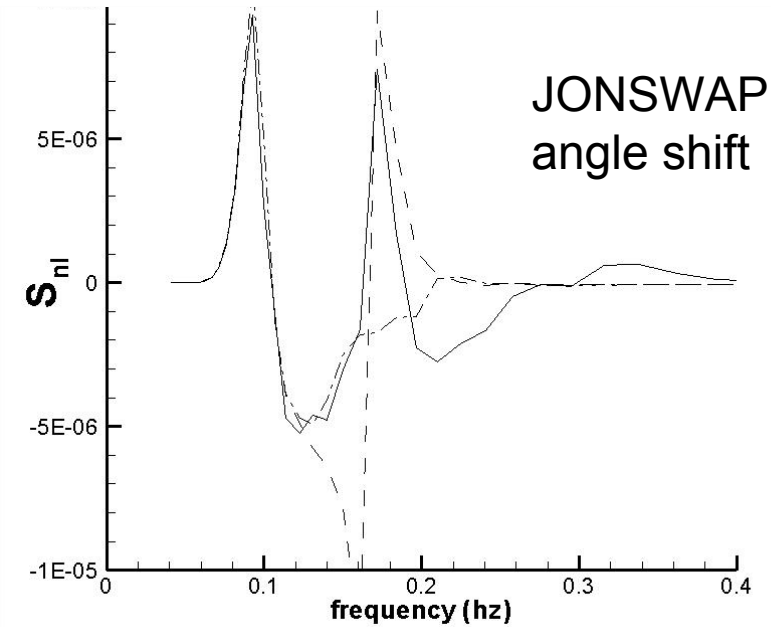

**Performance of TSA for JONSWAP spectra with selected peakednesses**
**Solid – full integral**
**Dot-dash – B-scale alone**
**Dashed - TSA**




JONSWAP with negative Gaussian perturbation

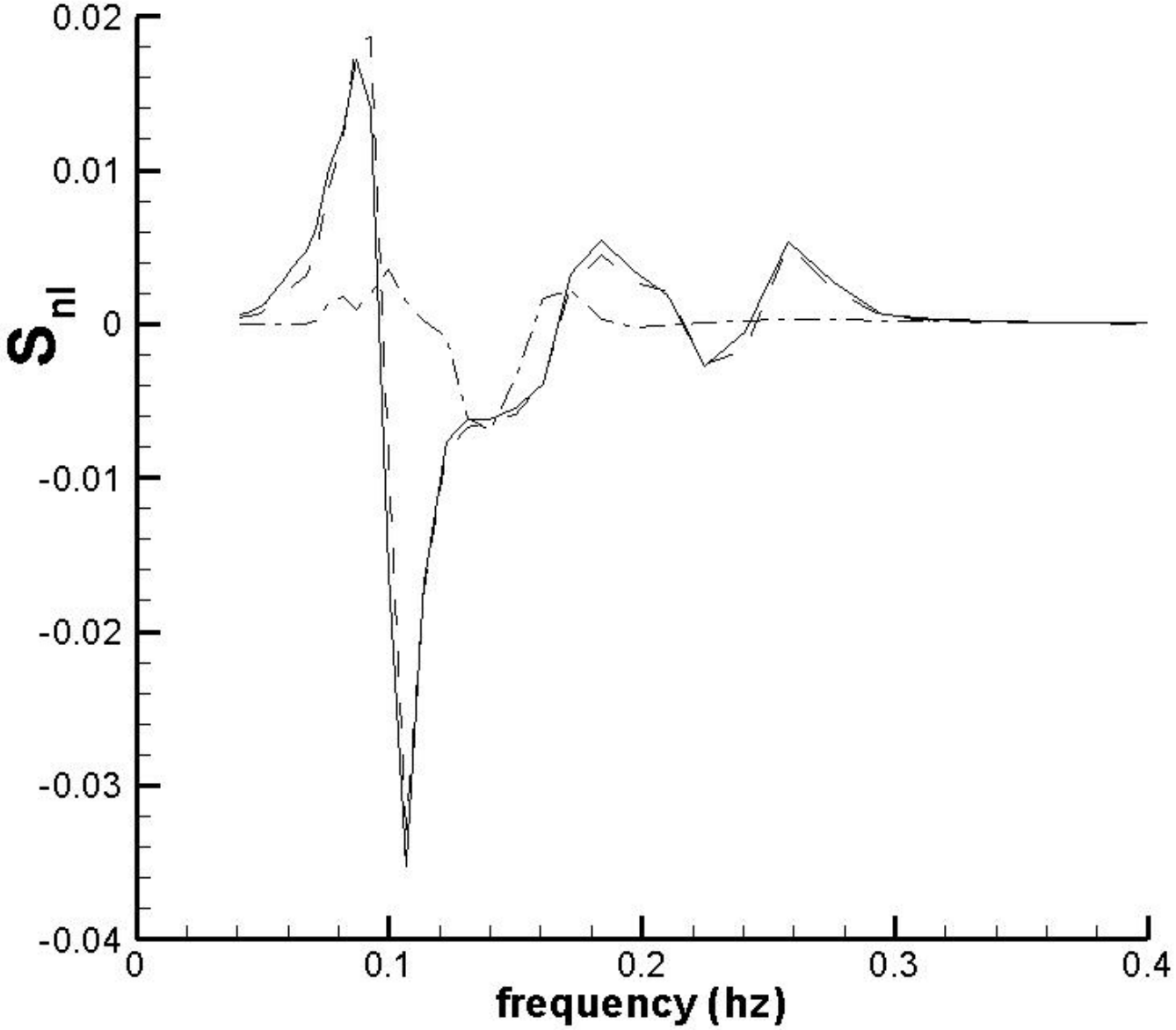



JONSWAP with positive Gaussian perturbation



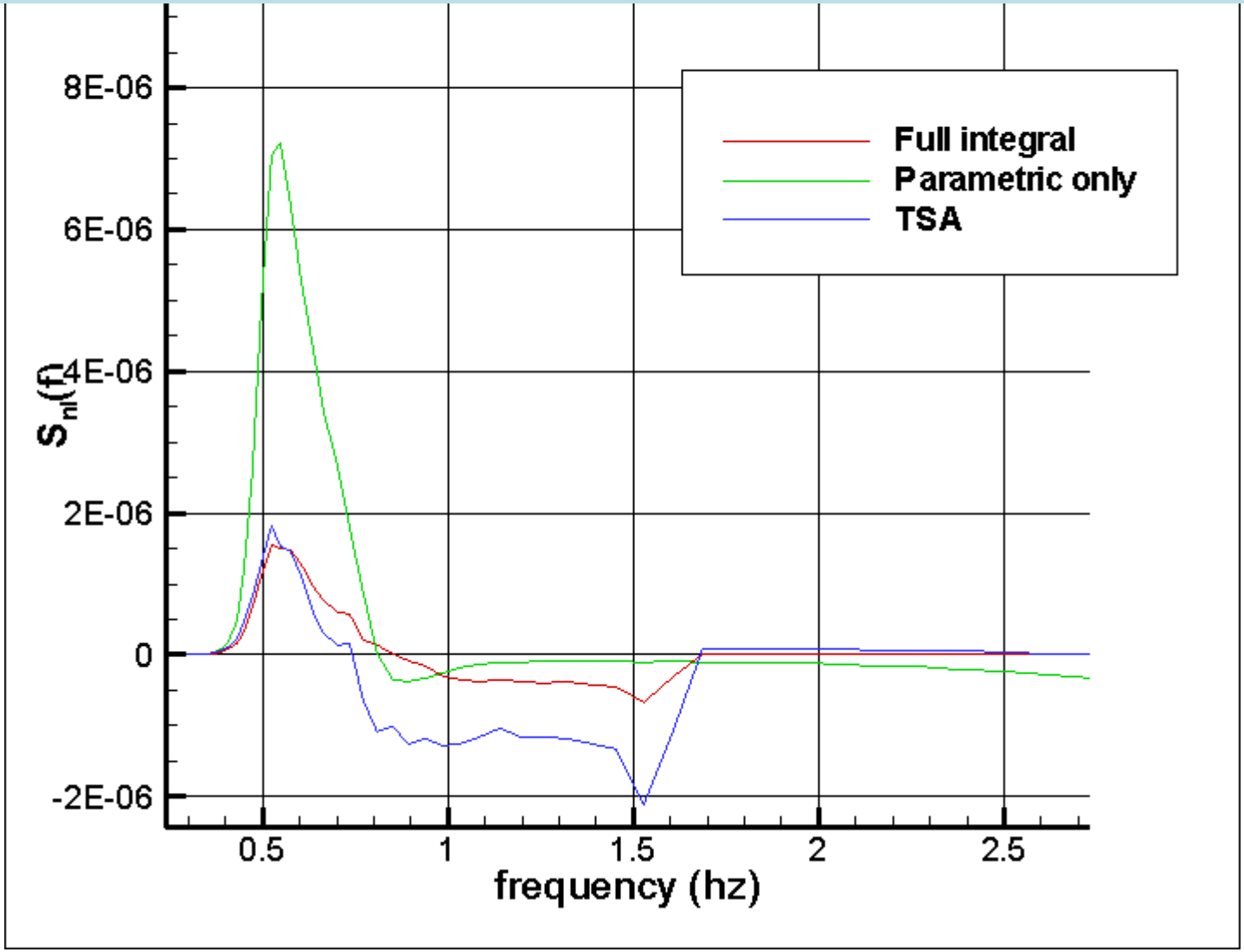

JONSWAP with 40-degree angle shift

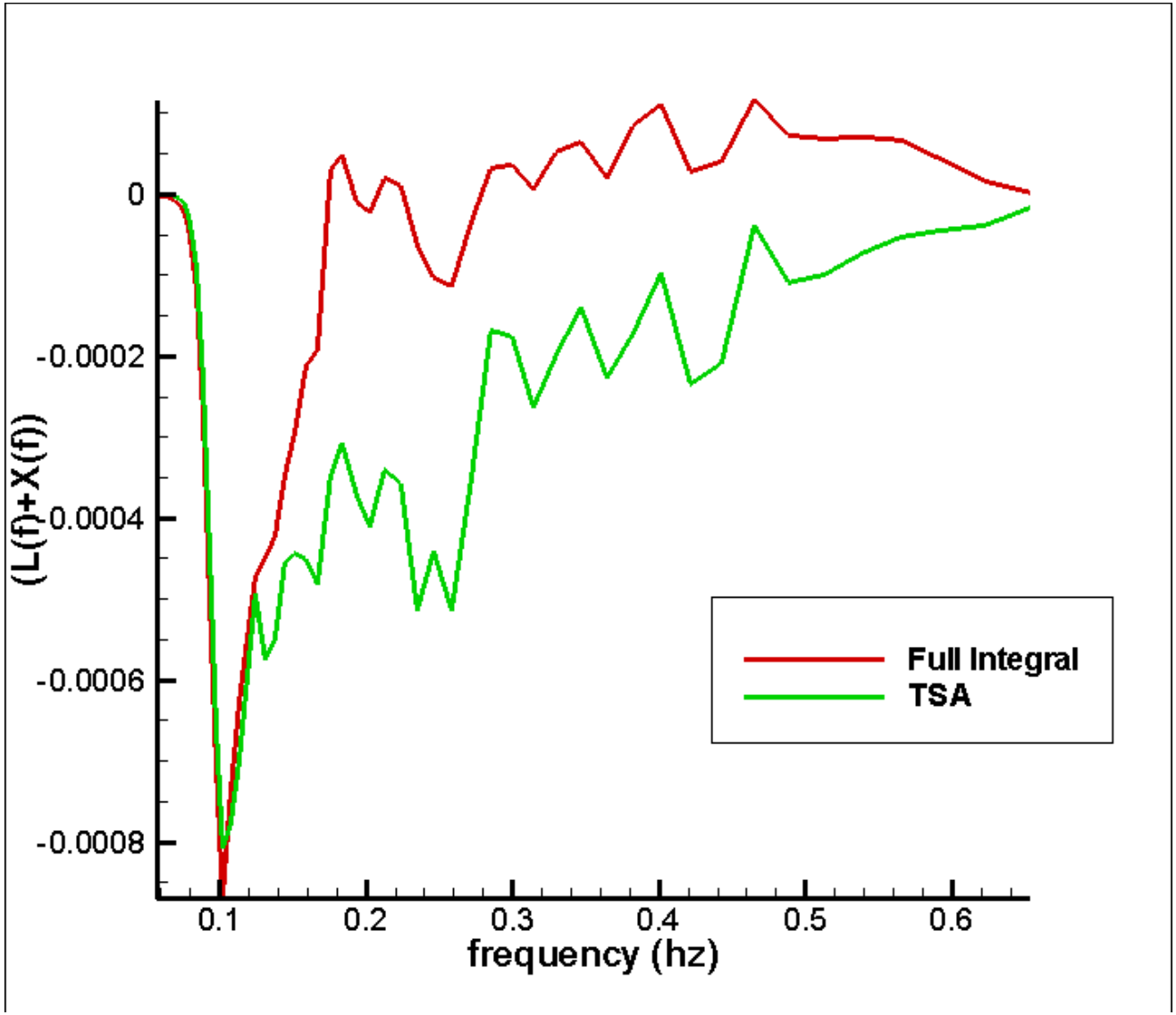
**Performance of TSA for more complicated spectra**
**Same line patterns as previous**




**Comparison of TSA (dashed) and DIA (dot-dash) to full integral (solid) for finite-depth case ($k_p h = 0.7$).**

**Example of actual spectrum with crude parameterization.**
**In this case, the spectral shape is highly variable in terms of its angular distribution and the peak shape is not well approximated.**






**Comparison of the L+X contributions to the equivalent contributions in the full integral provides a good measure of the effectiveness of the second-scale introduction in the TSA**

**The TSA**

- **conserves constants of motion (action, energy, momentum)**
- **has its basis in the correct integral (important for complex cases)**
- **retains the number of degrees of freedom as the modeled spectrum**
- **is not limited to kph ≥ 1**

**AND**

- **is much more efficient than the full integral**

**Number of mathematical operations is**

**about 3xDIA for 1 quadraplet (about 1.5 if two sets of q's are used)**
**<u>BUT: DIA's instability makes it unstable for moderate time steps!</u>**

**about 1/250 of the time of the Full Integral**

# CONCLUSIONS

- DIA has extreme difficulty in reproducing Snl since it does not represent the full set of 4-wave interactions.

- Although DIA is calibrated to provide similar energy transfer to the Low-frequency region of the spectrum, the calibration is only locally valid (near $\gamma$=3.3).

- The parametric extension of the DIA to shallow water does not capture essential elements of nonlinear energy fluxes and their effects on spectral shape in coastal waters

- The TSA appears to be a more accurate alternative to the DIA for both deep water and shallow water cases

- The initial TSA can be extended to improve the B-scale treatment which should improve its accuracy for operational applications.