COMPUTATIONS AND DATA ANALYSIS OF VERY NONLINEAR, DIRECTIONALLY SPREAD, SHALLOW WATER WAVES

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I. INTRODUCTION

Second order surface wave theory is an important tool of modern engineering, often used for computing the influence of nonlinear effects on simulations of random, directionally spread wave trains (Forristall [2000]). Implicit in the method is that the second-order Stokes corrections must remain small. While this approach works well for many cases, it is clear that Stokes harmonics of much higher order can occur, particularly in shallow water where linear superposition is no longer tenable [Smith and Vincent, 1995]. The focus of the present paper is to address how Stokes corrections can be implemented *to all orders* in the theory, in the shallow water approximation, by applying the method of the *inverse scattering transform* (IST).

To this end it is also important to theoretically address computation of directional spectra, to arbitrary order in nonlinearity, for shallow water waves. We begin by considering second-order water wave theory (Sharma and Dean, [1989], Longuet-Higgins [1963]) from the point of view of the IST in the shallow water limit. We discuss shallow water theory from the point of view of the Korteweg-deVries (KdV) equation (which describes unidirectional waves) and the Kadomtsev-Petviashvili (KP) equation (which includes directional spreading, albeit for relatively small angles). By comparing the IST formulation for shallow water theory with second order theory we find an expression for the *nonlinear directional spectrum* (Riemann matrix) in terms of wave number and/or frequency and direction. These results allow us to compute to all orders the nonlinear effects inherent in shallow water, nonlinear wave dynamics.

This new approach, based upon IST, has several features, some of which we now summarize. The method: (1) Connects second order theory with the infinite-order, nonlinear Fourier analysis formulation for the KdV and KP equations and higher order extensions. (2) Allows fully nonlinear directional spectra to be computed from data. (3) Computes nonlinear directional spectra, not just in terms of ordinary sine waves, but also in terms of solitons and Stokes waves, i.e. the natural nonlinear basis functions of the IST. (4) Allows previously developed statistical approaches for second order theory to be applied to inverse scattering formulations. Extension to third order is straightforward, for example.

A brief outline of the paper is as follows. We give a discussion of shallow water theory in Section II. An overview of the inverse scattering transform for KdV and KP is provided in Section IIIA, B. Then in Section IIIC we discuss the truncation of the IST formulation to second order. Finally in Section IV we discuss details for the computation of nonlinear, directional wave trains from directional spectra. Then in Section V we give a numerical example of the approach. In the summary we discuss the range of applicability of second order theory by comparing to the IST. We use the Ursell number to the characterize the nonlinearity of the waves.

II. SECOND ORDER THEORY

Second order wave theory (Sharma and Dean [1979]; Forristall [2000]) has the form

$$\eta(x,y,t) = \sum_{n=1}^{N} a_n \cos X_n + \\ +\tfrac{1}{4}\sum_{j=1}^{N}\sum_{k=1}^{N} a_j a_k K_{ij}^{+} \cos(X_j + X_k) + \\ +\tfrac{1}{4}\sum_{j=1}^{N}\sum_{k=1}^{N} a_j a_k K_{ij}^{-} \cos(X_j - X_k) \tag{1}$$

where

$$X_n = \mathbf{k}_n \cdot \mathbf{x} - \omega_n t + \phi_n \tag{2}$$

Here the $\mathbf{k}_n = [k_{nx}, k_{ny}] = [k_n, l_n]$ are the wave number vectors in the horizontal plane, $\mathbf{x} = [x, y]$ is the position vector, the ω_n are the frequencies and the ϕ_n are the phases.

The single sum on the right hand side of (1) is the usual linear Fourier series which contains directional spreading, while the other two terms are the nonlinear, Stokes-like corrections to second order. Two Stokes corrections are made, one is related to the sums of wave numbers $(X_j + X_k)$ and frequencies and the second is related to the differences $(X_j - X_k)$.

The coefficients in the Stokes second order corrections have the form:

$$K_{ij}^{-} = \left[D_{ij}^{-} - (\mathbf{k}_i \cdot \mathbf{k}_j + R_i R_j)\right](R_i R_j)^{-1/2} + R_i + R_j \tag{3}$$

$$K_{ij}^{+} = \left[D_{ij}^{+} - (\mathbf{k}_i \cdot \mathbf{k}_j - R_i R_j)\right](R_i R_j)^{-1/2} + R_i + R_j \tag{4}$$

$$D_{ij}^{-} = \frac{\left(\sqrt{R_i}-\sqrt{R_j}\right)\left[\sqrt{R_j}\left(\mathbf{k}_i^2-R_i^2\right)-\sqrt{R_i}\left(\mathbf{k}_j^2-R_j^2\right)\right]}{\left(\sqrt{R_i}-\sqrt{R_j}\right)^2-k_{ij}^{-}\tanh k_{ij}^{-}h} + \frac{2\left(\sqrt{R_i}-\sqrt{R_j}\right)^2(\mathbf{k}_i\cdot\mathbf{k}_j+R_iR_j)}{\left(\sqrt{R_i}-\sqrt{R_j}\right)^2-k_{ij}^{-}\tanh k_{ij}^{-}h} \tag{5}$$

$$D_{ij}^{+} = \frac{\left(\sqrt{R_i}+\sqrt{R_j}\right)\left[\sqrt{R_i}\left(\mathbf{k}_i^2-R_i^2\right)+\sqrt{R_j}\left(\mathbf{k}_j^2-R_i^2\right)\right]}{\left(\sqrt{R_i}+\sqrt{R_j}\right)^2-k_{ij}^{+}\tanh k_{ij}^{+}h} + \frac{2\left(\sqrt{R_i}+\sqrt{R_j}\right)^2(\mathbf{k}_i\cdot\mathbf{k}_j+R_iR_j)}{\left(\sqrt{R_i}+\sqrt{R_j}\right)^2-k_{ij}^{+}\tanh k_{ij}^{+}h} \tag{6}$$

where

$$k_{ij}^{-} = |\mathbf{k}_i - \mathbf{k}_j| \tag{7}$$

$$k_{ij}^{+} = |\mathbf{k}_i + \mathbf{k}_j| \tag{8}$$

$$R_i = |\mathbf{k}_i| \tanh(|\mathbf{k}_i|\, h) = \omega_i^2/g \tag{9}$$

Aside from a minor typo the above theory reduces to that of Longuet-Higgins [1963] for infinite water depth.

III. THE INVERSE SCATTERING TRANSFORM

In shallow water it is often convenient to address the physics in terms of the Kortweg-deVries equation (KdV, unidirectional propagation) and the Kadomtsev-Petviashvili equation (KP, which contains directional spreading). The cases are distinct enough to consider separately, although the KdV equation is contained within the KP formulation when there is no directional spreading, i.e. when the y component wave number $l_n = 0$. We first discuss the case for KdV.

A. The KdV Equation

The KdV equation (Korteweg and deVries [1895]) describes unidirectional wave propagation in shallow water. It is given by

$$\eta_t + c_o\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = 0 \tag{10}$$

where $c_o = \sqrt{gh}$, $\alpha = 3c_o/2h$, $\beta = c_o h^2/6$ and h is the water depth. We assume periodic boundary conditions, $\eta(x,t) = \eta(x+L,t)$. The non-singular (physical) solutions of KdV are written in terms of multidimensional Fourier series (Riemann theta functions) given by (Dubrovin and Novikov [1976]; Its and Matveev [1976]):

$$\eta(x,t) = \frac{2}{\lambda}\partial_{xx}\ln\theta(x,t) = \frac{2}{\lambda}\left(\frac{\theta\theta_{xx}-\theta_x^2}{\theta^2}\right) \tag{11}$$

where $\lambda = \alpha/6\beta = 3/2h^3$. Here the Riemann theta function has the form:

$$\theta(x,t) = \sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\cdots\sum_{m_N=-\infty}^{\infty} \times e^{-\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} m_j m_k B_{jk}}\, e^{i\sum_{j=1}^{N} m_j k_j x - i\sum_{j=1}^{N} m_j\omega_j t + i\sum_{j=1}^{N} m_j\phi_j} \tag{12}$$

The B_{jk} are the elements of the Riemann matrix, k_j are the IST wave numbers, ω_j are the frequencies and ϕ_j are the phases in the IST spectrum. By abuse of notation the IST parameters ω_n, ϕ_n are not those of linear Fourier analysis. Rules for the determination of these parameters are discussed in Section IV. Note that the Riemann matrix plays the role of a nonlinear spectrum; it reduces to the linear Fourier transform in the small-amplitude limit, i.e. when the off-diagonal terms, B_{mn}, become small relative to the diagonal terms B_{nn}.

Due to the periodic boundary conditions we have the following rule for computing the wave numbers: $k_j = 2\pi j/L$, where L is the period of the wave train; this coincides with linear Fourier analysis. The number of nested sums in the theta function, N, provides the number of degrees of freedom or components in the IST spectrum. Because of the structure of the theta function the multiple (nested) summations occur over *all possible sums and differences of the wave numbers and frequencies.* Thus we say that the solution to the KdV equation is computable to all orders.

As just mentioned, the spectrum for the KdV equation is a matrix (the Riemann matrix) rather than a vector (as for the linear Fourier transform). The elements of the diagonal of the Riemann matrix correspond to cnoidal waves (Wiegel [1964]), which are the N nonlinear modes of the KdV equation (Osborne [1995]); these contrast with the sine wave modes of linear Fourier analysis. All solutions to KdV may be written as a *linear sum of cnoidal waves plus nonlinear interactions.* The nonlinear interactions are contained in the off-diagonal terms of the Riemann matrix. Since the cnoidal waves depend on their modulus, m, which varies between 0 and 1, the nonlinear modes of KdV can be sine waves ($m \sim 0$), Stokes waves ($m \sim 0.5$) and solitons ($m \sim 1$); as the modulus is increased from 0 to 1 the cnoidal wave varies smoothly from a sine wave to a Stokes wave to a soliton. The more nonlinear modes the KdV equation has in a particular application, the more terms we need to sum in the theta function, i.e. $\sim 10^N$. Theta functions are therefore a challenge to compute (Osborne [1995] [2002]).

To solve the Cauchy problem for the KdV equation we need to find the solution to the equation, $\eta(x,t)$, for a given initial condition $\eta(x,0)$. To do this we need to be able to compute the B_{jk}, k_n, ω_j and ϕ_j from $\eta(x,0)$. A practical procedure for doing this is given elsewhere (Osborne [1995]). A major goal of the present paper is to provide estimates of the B_{jk} on a physical basis, by comparing the IST to second order wave theory, see Section IIIC.

B. The Kadomtsev-Petviashvili Equation

The Kadomtsev-Petviashvili equation (Kadomtsev-Petviashvili [1973]), which describes the nonlinear dynamics of directionally spread surface waves in shallow water, is written:

$$(\eta_t + c_o\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx})_x + \gamma\eta_{yy} = 0 \qquad (13)$$

where the coefficients c_o, α, β are the same as in the KdV equation above and $\gamma = c_o/2$. The wave field also depends on the y coordinate, $\eta(x,y,t)$. Assuming periodic boundary conditions, $\eta(x,y,t) = \eta(x+L_x, y+L_y, t)$, the solution to KP has the form:

$$\eta(x,y,t) = \tfrac{2}{\lambda}\partial_{xx}\ln\theta(x,y,t) =$$
$$= \tfrac{2}{\lambda}\left[\left(\theta(x,y,t)\theta_{xx}(x,y,t) - \theta_x^2(x,y,t)\right)/\theta^2(x,y,t)\right] \qquad (14)$$

where

$$\theta(x,y,t) = \sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\cdots\sum_{m_N=-\infty}^{\infty} e^{-\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} m_j m_k B_{jk}}$$
$$\times e^{i\sum_{j=1}^{N} m_j k_j x + i\sum_{j=1}^{N} m_j l_j y - i\sum_{j=1}^{N} m_j \omega_j t + i\sum_{j=1}^{N} m_j\phi_j} \qquad (15)$$

The x and y wave numbers are given by $k_j = 2\pi j/L_x$ and $l_j = 2\pi j/L_y$ for periodic boundary conditions. As before the number of nested sums in the theta function, N, provides the number of nonlinear modes or *cnoidal waves* in the IST spectrum. Here each *diagonal element*, B_{nn}, corresponds to a wave number pair $[k_n, l_n]$ which specifies the *direction of a particular cnoidal wave.* Interactions among components at these wave numbers are contained in the off-diagonal terms, $B_{mn}, m \neq n$, of the Riemann matrix. Thus the general periodic solutions of the KP equation consist of a *linear superposition of directionally spread cnoidal waves plus nonlinear interactions among them.*

The explicit expression for the solution of the KP equation which best illustrates these ideas can be obtained by combing (14), (15) [11]:

$$\eta(x,y,t) =$$
$$\sum_{n=1}^{N} a_n cn^2\{[K(m)/\pi](k_n x + l_n y - \omega_n t + \phi_n | m_n)\} +$$
$$+ \eta_{\rm int}(x,y,t) \qquad (16)$$

Eq. (16) is physically interpreted as the sum of N cnoidal-wave spectral components (the basis functions of nonlinear spectral theory) plus nonlinear interactions among the cnoidal waves ($\eta_{\rm int}(x,y,t)$, see [11] for an explicit expression for the KdV equation). Here $cn(k_n x + l_n y - \omega_n t|m_n)$ is the classical Jabcobian elliptic function [1] and m_n is its modulus ($0 \leqslant m_n < 1$). The cnoidal wave amplitudes, a_n, are graphed as a function of frequency and direction; this graph is known as the *nonlinear directional spectrum.* The a_n are computed from the respective nomes, $q_n = \exp(-B_{nn}/2)$, using the diagonal elements of the Riemann matrix, B_{nn} [1]:

$$a_n = \frac{4k_n^2}{\lambda}\sum_{m=1}^{\infty}\frac{mq_n^m}{1-q_n^{2m}} \qquad (17)$$

The *Ursell number* of a cnoidal wave, $U_n = 3a_n/4k^2h^3$, is given in terms of the modulus, m_n, by:

$$m_n K^2(m_n) = 6\pi^2 a_n/4k_n^2 h^3 = 2\pi^2 U_n \qquad (18)$$

$K(m)$ is the real quarter-period of the elliptic function. Therefore the modulus, m_n, and the Ursell number, U_n, are two equally good parameters for describing the (Stokes-type) nonlinearity in a cnoidal wave.

The *soliton* occurs in the limit as $m_n \to 1$ and therefore in the numerical work below we will consider a *cnoidal wave spectral component to be a soliton* provided that m_n is sufficiently near one. A *soliton gas* [15] is here defined as a wave train for which a large number of the diagonal elements of the Riemann matrix are so small that their modulus is near one, i.e. the spectrum is *soliton dominated.* In highly nonlinear cases soliton dominance typically occurs in a band from low to high frequency extending across most of the spectrum. For example, from an experimental point of view we speak of data which is soliton dominated as a soliton gas.

C. Truncation of IST Theory to Second Order Theory

As pointed out above the theta function contains all possible sums and differences of the wave numbers (and frequencies), including pairs of wave numbers, triples, quadruples, etc. Typically one sums over $\sim 10^N$ terms, where N is the number of cnoidal waves in the spectrum. However, second order theory provides only sums and differences of *pairs of wave numbers.* Therefore to make a comparison between the two theories we need to truncate the theta series to only sums and differences of pairs of wave numbers. To this end we take a particular partial sum of the theta function, summing only terms from -1 to 1, rather than $-\infty$ to ∞. Stated this way the truncation seems quite drastic, and indeed this is the case, but as we shall see this procedure leads directly to second order theory in shallow water. Hence we take:

$$\theta(x,y,t) \cong \sum_{m_1=-1}^{1} \sum_{m_2=-1}^{1} \dots \sum_{m_N=-1}^{1} \times e^{-\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} m_j m_k B_{jk}} e^{i\sum_{j=1}^{N} m_j X_j} \tag{19}$$

where $X_n = k_n x - \omega_n t + \phi_n$ for the KdV equation and $X_n = k_n x + l_n y - \omega_n t + \phi_n$ for the KP equation. Expanding this expression and truncating at second order we have:

$$\begin{aligned}\theta(x,y,t) \cong\ & 1 + 2\sum_{n=1}^{N} e^{-\frac{1}{2}B_{nn}} \cos[X_n] + \\ & +2\sum_{m=1}^{N}\sum_{n=1}^{N} e^{-\frac{1}{2}(B_{mm}+2B_{mn}+B_{nn})} \cos[X_m + X_n] + \\ & +2\sum_{m=1}^{N}\sum_{n=1}^{N} e^{-\frac{1}{2}(B_{mm}-2B_{mn}+B_{nn})} \cos[X_m - X_n] + \dots\end{aligned} \tag{20}$$

Higher order terms include higher harmonics and triples, quadruples, etc. of the wave numbers, but we neglect these higher sums and differences of the wave number components in order to compare directly to second order theory. Assume the second order terms are small and write $\ln(1+a) \cong a - a^2/2$, which gives

$$\begin{aligned}\ln\theta(x,y,t) \cong\ & 2\sum_{n=1}^{N} q_n \cos[X_n] + \\ & + \sum_{m=1}^{N}\sum_{n=1}^{N} G_{mn}^{+} q_m q_n \cos[X_m + X_n] + \\ & + \sum_{m=1}^{N}\sum_{n=1}^{N} G_{mn}^{-} q_m q_n \cos[X_m - X_n] + \dots\end{aligned} \tag{21}$$

where $q_n = \exp(-B_{nn}/2)$ is the nome and

$$G_{mn}^{+} = 2e^{-B_{mn}} - 1 \tag{22}$$

$$G_{mn}^{-} = 2e^{B_{mn}} - 1 \tag{23}$$

The second spatial derivative of this expression gives the surface elevation to second order

$$\begin{aligned}\eta(x,y,t) \cong\ & \tfrac{4}{\lambda}\sum_{n=1}^{N} k_n^2 q_n \cos[X_n] + \\ & +\tfrac{2}{\lambda}\sum_{m=1}^{N}\sum_{n=1}^{N} (k_m + k_n)^2 G_{mn}^{+} q_m q_n \cos[X_m + X_n] + \\ & +\tfrac{2}{\lambda}\sum_{m=1}^{N}\sum_{n=1}^{N} (k_m - k_n)^2 G_{mn}^{-} q_m q_n \cos[X_m - X_n] + \dots\end{aligned} \tag{24}$$

The first term in the above expression (24) is just the usual linear Fourier directional spectrum and is formally equivalent to the first term in second order theory (1). Each of the amplitude terms, a_n, in the linear Fourier series corresponds to a single wave number pair k_n, l_n in the wave number plane in two dimensions and therefore a_n vs k_n, l_n constitutes the linear directional spectrum, a result which is well known. The expression (24) can be compared to second-order theory to establish the Riemann matrix in terms of the wave number and frequency. We thus compare (1) to (24) and find

$$a_n = \frac{4}{\lambda} k_n^2 e^{-\frac{1}{2}B_{nn}} = \frac{4}{\lambda} k_n^2 q_n \tag{25}$$

$$a_m a_n K_{mn}^{+} = \frac{8}{\lambda}(k_m + k_n)^2 G_{mn}^{+} q_m q_n \tag{26}$$

$$a_m a_n K_{mn}^{-} = \frac{8}{\lambda}(k_m - k_n)^2 G_{mn}^{-} q_m q_n \tag{27}$$

The first of these expressions (25) can be inverted and provides us with the diagonal elements of the Riemann matrix, B_{nn}. We thus see that, to second order, the diagonal elements B_{nn} are directly related to the linear Fourier coefficients, a_n. Additionally, given the expressions for K_{mn}^{+} and K_{mn}^{-} in shallow water theory (3), (4), the second and third of the above two equations can be solved for the off-diagonal terms, i.e. from (26) and (27) we find:

$$B_{mn} = -\frac{1}{2}\ln\left[\frac{J_{mn}^{+}}{J_{mn}^{-}}\left(\frac{k_m - k_n}{k_m + k_n}\right)^2\right]; m \neq n \tag{28}$$

where

$$J_{mn}^{+} = K_{mn}^{+} + \frac{\lambda}{2}\left(\frac{k_m + k_n}{k_m k_n}\right)^2 \tag{29}$$

$$J_{mn}^{-} = K_{mn}^{-} + \frac{\lambda}{2}\left(\frac{k_m - k_n}{k_m k_n}\right)^2 \tag{30}$$

This is the *leading order expression for the Riemann matrix for directionally spread wave trains.* Note that, in this approximation, the off-diagonal terms (28) depend only on the wave number and frequency, not on the amplitudes of the waves. Instead the diagonal terms (25) also depend directly on the wave amplitudes.

In the case of the KdV equation we have the shallow water limit, for which there is no directional spreading and we find:

$$\frac{J_{mn}^{+}}{J_{mn}^{-}} \cong 1 \tag{31}$$

This then gives the following spectral matrix:

$$B_{nn} = -2\ln\left(\frac{\lambda a_n}{4k_n^2}\right) \tag{32}$$

$$B_{mn} = -\frac{1}{2} \ln \left[\left(\frac{k_m - k_n}{k_m + k_n} \right)^2 \right] ; m \neq n \tag{33}$$

These results agree with those of Section IV which discuss the Schottky uniformization approach for obtaining periodic solutions to the KP equation. Eqs. (32) and (33) are the fundamental leading order approximations for the Riemann matrix for unidirectional shallow water waves.

Other useful relations are (combine (25) with (26), (27)):

$$K^+_{mn} = \frac{\lambda}{2} \left(\frac{k_m + k_n}{k_m k_n} \right)^2 G^+_{mn} ; m \neq n \tag{34}$$

$$K^-_{mn} = \frac{\lambda}{2} \left(\frac{k_m - k_n}{k_m k_n} \right)^2 G^-_{mn} ; m \neq n \tag{35}$$

Taking the ratio of these last two equations allows us to solve for the off-diagonal terms, a result which reproduces (28) above. Taking instead the product of equations (34), (35) yields

$$K^+_{mn} K^-_{mn} = \left(\frac{\lambda}{4} \right)^2 \left[\frac{(k_m + k_n)(k_m - k_n)}{k_m^2 k_n^2} \right]^2 G^+_{mn} G^-_{mn} \tag{36}$$

The interaction (skewness) kernal is a useful measure of the second order interactions as discussed by Forristall [2000]:

$$\begin{aligned} &\tfrac{1}{4} \left(K^+_{mn} + K^-_{mn} \right) = \\ &\qquad = \tfrac{\lambda}{8} \left[\tfrac{(k_m + k_n)^2 G^+_{mn} + (k_m - k_n)^2 G^-_{mn}}{k_m^2 k_n^2} \right] \end{aligned} \tag{37}$$

Note that this expression has been written in terms of the Riemann matrix since $G^{\pm}_{mn} = G^{\pm}_{mn}(B_{mn})$.

In all of the formulas above it is worth noting that for periodic boundary conditions (often used for data analysis and for numerical modeling purposes) we use the commensurable wave numbers: $k_n = 2\pi n/L$. In this case the following expression is seen to simplify:

$$\frac{k_m - k_n}{k_m + k_n} = \frac{m - n}{m + n} \tag{38}$$

IV. COMPUTATION OF DIRECTIONALLY SPREAD WAVE TRAINS IN SHALLOW WATER

A procedure known as *Schottky uniformization* allows us to get all non-singlular, periodic solutions of the KP equation in a straightforward manner (Baker [1898]; Bobenko [1989]; Osborne, unpublished). Here we review the method and discuss results important to the main body of this paper.

For (14, 15) to constitute a solution to the KP equation one must compute the following formulas for the wave numbers, frequencies and Riemann matrix in terms of *Poincaré series* as a function of *uniformization parameters* $A_n, \mu_n, n = 1, N$. To give a physical interpretation of these parameters it is enough to remember that at leading order the μ_n are related to the diagonal elements of the Riemann matrix (or the amplitudes of the cnoidal waves) and the A_n (complex numbers) are related to the wave numbers $k_n, l_n, n = 1, N$ (real numbers); the exact relationships are now given among these parameters for the wavenumbers, frequencies and Riemann matrix. In the formulas below the σ are linear fractional transformations in terms of the uniformization parameters, $\sigma_n = \sigma_n(A_n, \mu_n)$. The summations given below are group theoretic, use standard group theory notation, and include the identity, $\sigma = I$.

The diagonal elements of the Riemann matrix are:

$$B_{nn} = \ln \mu_n + \sum_{\sigma \in G_n \backslash G/G_n, \sigma \neq I} \ln \left[\frac{(A_n^* - \sigma A_n^*)(A_n - \sigma A_n)}{(A_n^* - \sigma A_n)(A_n - \sigma A_n^*)} \right] \tag{39}$$

The off-diagonal elements have the form:

$$B_{mn} = \tfrac{1}{2} \sum_{\sigma \in G_n \backslash G/G_n} \ln \left[\tfrac{(A_m^* - \sigma A_n^*)(A_m - \sigma A_n)}{(A_m^* - \sigma A_n)(A_m - \sigma A_n^*)} \right] \quad m \neq n \tag{40}$$

or alternatively

$$\begin{aligned} B_{mn} = &-\tfrac{1}{2} \ln \left[\tfrac{(A_m^* - A_n^*)(A_m - A_n)}{(A_m^* - A_n)(A_m - A_n^*)} \right] + \\ &-\tfrac{1}{2} \sum_{\sigma \in G_n \backslash G/G_n, \sigma \neq I} \ln \left[\tfrac{(A_m^* - \sigma A_n^*)(A_m - \sigma A_n)}{(A_m^* - \sigma A_n)(A_m - \sigma A_n^*)} \right] \\ &m \neq n \end{aligned} \tag{41}$$

In the last equation we have brought out the term for $\sigma = I$, the identity, and displayed it before the group theoretic summation; note that this latter summation now excludes the identity. The wave numbers have the form:

$$k_n = \sum_{\sigma \in G/G_n} (\sigma A_n - \sigma A_n^*) \tag{42}$$

$$l_n = h \sum_{\sigma \in G/G_n} \left[(\sigma A_n)^2 - (\sigma A_n^*)^2 \right] \tag{43}$$

The frequency is

$$\omega_n = c_o k_o - 4\beta \sum_{\sigma \in G/G_n} \left[(\sigma A_n)^3 - (\sigma A_n^*)^3 \right] \tag{44}$$

Separating the identity term from the others leads to alternative forms for (42)-(44):

$$k_n = (A_n - A_n^*) + \sum_{\sigma \in G/G_n, \sigma \neq I} (\sigma A_n - \sigma A_n^*) \tag{45}$$

$$l_n = h\left[A_n^2 - A_n^{*2}\right] + h \sum_{\sigma\in G/G_n,\sigma\neq I} \left[(\sigma A_n)^2 - (\sigma A_n^*)^2\right] \tag{46}$$

$$\omega_n = c_o k_o - 4\beta\left[A_n^3 - A_n^{*3}\right] - \\ -4\beta \sum_{\sigma\in G/G_n,\sigma\neq I} \left[(\sigma A_n)^3 - (\sigma A_n^*)^3\right] \tag{47}$$

It is not hard to show that only the leading order terms contribute when the waves have small amplitude. Thus eqs. (41), (45)-(47) have a small amplitude limit: These are formulas which are important for the comparison with second-order theory given herein:

$$B_{nn} \cong \ln \mu_n \tag{48}$$

$$B_{mn} \cong \frac{1}{2}\ln\left[\frac{(A_m^* - A_n^*)(A_m - A_n)}{(A_m^* - A_n)(A_m - A_n^*)}\right] \tag{49}$$

$$k_n \cong A_n - A_n^* \tag{50}$$

$$l_n \cong h\left(A_n^2 - A_n^{*2}\right) \tag{51}$$

$$\omega_n = c_o k_n - 4\beta\left[A_n^3 - A_n^{*3}\right] \tag{52}$$

Assuming that k_n, l_n are given wave numbers in the plane we can easily solve (50) and (51), approximately, for A_n. Thus, to leading order the Schottky uniformization variables have the form

$$A_n = a_n + ib_n \simeq \frac{1}{2}\left(\frac{l_n}{hk_n} + ik_n\right) \tag{53}$$

where a_n is the real part of A_n and b_n is the imaginary part:

$$a_n = \frac{l_n}{2hk_n}; \qquad b_n = \frac{k_n}{2} \tag{54}$$

These are useful results because they point out the one-to-one relationship between the Schottky uniformization parameters ($A_n = a_n + ib_n$) and the wave number pair (k_n, l_n).

Insert A_n (53) into (49) to obtain the leading order expressions for the Riemann matrix:

$$B_{mn} \cong \frac{1}{2}\ln\left[\frac{(k_m - k_n)^2 + \left(\frac{k_n l_m - k_m l_n}{k_m k_n}\right)^2}{(k_m + k_n)^2 + \left(\frac{k_n l_m - k_m l_n}{k_m k_n}\right)^2}\right]; m \neq n \tag{55}$$

Note that when $l_n = 0$ this expression reduces to the shallow water approximation for (28). Another form, more compatible with eq. (28), is given by

$$B_{mn} \cong \tfrac{1}{2}\ln\left[\frac{(k_m-k_n)^2}{(k_m+k_n)^2}\left(\frac{1+\left(\frac{k_n l_m - k_m l_n}{k_m k_n (k_m - k_n)}\right)^2}{1+\left(\frac{k_n l_m - k_m l_n}{k_m k_n (k_m + k_n)}\right)^2}\right)\right] \tag{56}$$

$$m \neq n$$

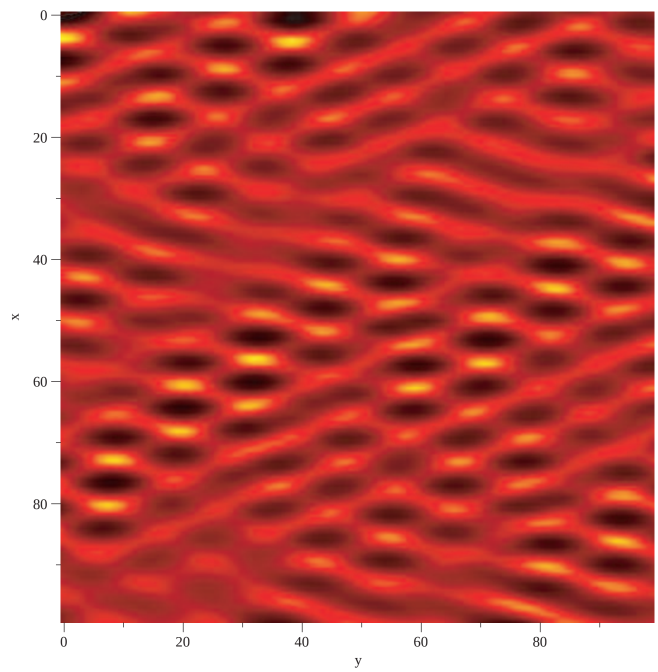

FIG. 1: Overhead view of directionally spread sea surface with 10 degrees of freedom using Riemann theta functions.

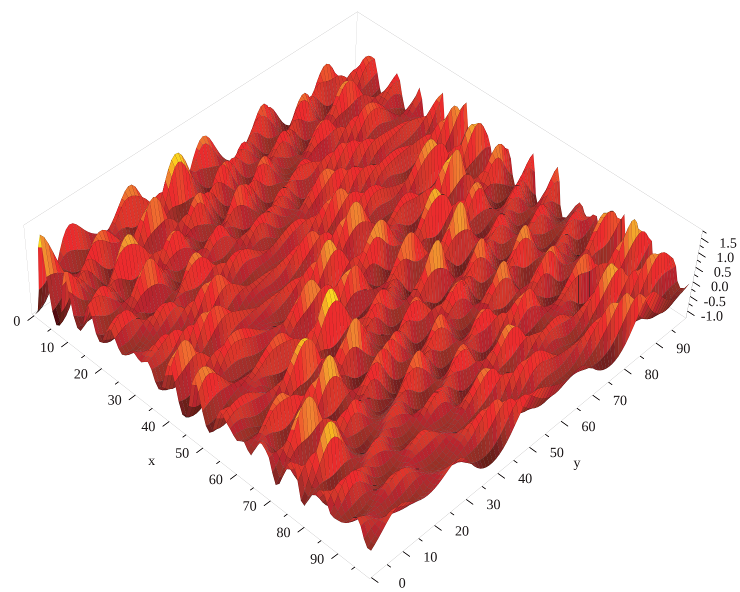

FIG. 2: Surface elevation from directionally spread sea with 10 degrees of freedom using Riemann theta functions.

which suggests

$$\frac{J_{mn}^+}{J_{mn}^-} = \frac{1 + \left(\frac{k_n l_m - k_m l_n}{k_m k_n (k_m - k_n)}\right)^2}{1 + \left(\frac{k_n l_m - k_m l_n}{k_m k_n (k_m + k_n)}\right)^2} \tag{57}$$

in the shallow water limit.

n	k_n	l_n	a_n	q_n	B_{nn}	m_n
1	0.04255	0.00000	.02634	.01065	9.0844	0.15669
2	0.04965	0.01986	.07254	.01824	8.0082	0.25322
3	0.04965	-0.01986	.30629	.08361	4.9630	0.74087
4	0.05674	0.01700	.33495	.07194	5.2639	0.68620
5	0.05674	-0.01700	.38295	.07515	5.1765	0.70226
6	0.06383	0.01596	.31352	.05411	5.8335	0.58069
7	0.06383	-0.0160	.34957	.05594	5.7669	0.59295
8	0.07093	0.01419	.15434	.02356	7.4966	0.31422
9	0.07093	-0.01419	.06503	.00937	9.3410	0.13921
10	0.07800	0.0000	.04344	.00523	10.0507	0.0803

TABLE I: Table of wave numbers, amplitudes, nomes, diagonal elements of the period matrix and moduli of the numerical simulation in Figs. 1 and 2

V. NUMERICAL PROCEDURES AND EXAMPLES

The numerical procedure for determining an N degree of freedom solution to the KP equation is to execute the following steps: (1) Pick the Riemann matrix diagonal elements, B_{nn}, and then compute the cnoidal wave amplitudes, a_n, by (17) ($1 \leq n \leq N$). Alternatively pick the amplitudes a_n and estimate the diagonal elements B_{nn} by inverting (17). Also choose a set of arbitrary phases, ϕ_n, one for each cnoidal wave. (2) Choose the set of wave numbers k_n, l_n corresponding to each of the cnoidal waves selected in step (1). (3) Given the B_{nn} and k_n, l_n use formulas (48) and (53) to estimate the uniformization parameters A_n, μ_n. (4) Treat (39), (42) and (43) as nonlinear equations which, given B_{nn} and k_n, l_n, one solves for the A_n, μ_n iteratively by standard techniques beginning with the starting values estimated in step (3). When the iterations give precise enough values of A_n, μ_n (i.e. when their values no longer change significantly between iterations) go to the next step. (4) Compute the off-diagonal elements of the Riemann matrix by (41) and the frequencies by (44). We now have available all elements of the period matrix, the wave numbers, the frequencies and the phases. This is sufficient information to use formulas (14), (15) to compute the solution to the KP equation.

We now show a numerical examples of a realization of a random, directionally spread wave train for the KP equation. This example is shown in Fig. 1 which shows the simulated sea state as seen from above. Fig. 2 shows the simulated sea surface for the simulation. The water depth is $h = 8$ m. There are 10 components in the spectrum which has $H_s = 0.4$ m.

VI. SUMMARY

We have presented a new approach which uses inverse scattering theory for the simulation of nonlinear, directional wave trains. To provide physical insight we have compared IST to second order theory and derived many of the results necessary for estimates of IST parameters directly from second order theory. We then computed a numerical example of a nonlinear, directionally spread random wave train which is highly nonlinear, far beyond the ability of second order theory to compute. We view these new methods as an efficient way for studying the behavior of highly nonlinear, directional surface waves in shallow water.

Acknowledgments

We gratefully acknowledge the support of the Army Corp of Engineers and the office of Naval Research.

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