Freak Waves as Nonlinear Stage of Stokes Wave Modulation Instability

Vladimir E. Zakharov[1,2,3,*], Alexander I. Dyachenko[2]

[1] Department of mathematics, University of Arizona, Ticson AZ 857201

[2] Landau Institute for Theoretical Physics, 2 Kosygin str. Moscow, 117334, Russia

[3] Waves and Solitons Ltd.

[*] e-mail: zakharov@math.arizona.edu

1 Introduction

Waves of extremely large size, alternatively called freak, rogue or giant waves are a well-documented hazard for mariners (see, for instance Smith 1976, Dean 1990, Chase 2003). These waves are responsible for loss of many ships and many human lives. Freak waves could appear in any place of the world ocean (see Earle 1975, Mori 2002, Divinsky et al 2004); however, in some regions they are more probable than in others. One of the regions where freak waves are especially frequent is the Agulhas current of the South-East coast of South Africa (see Gerber 1996, Gutshabas and Lavrenov 1986, Irvine and Tilley 1988, Lavrenov 1998, Mallory 1974). Peregrine in 1976 suggested that in areas of strong current such as the Agulhas, giant waves could be produced when wave action is concentrated by reflection into a caustic region. According to this theory, a variable current acts analogously to an optic lens to focus wave action. The caustic theory of freak waves was supported since this time by works of many authors. Among them Smith (1976), Gutshabas (1986), Irvine and Tilley (1988), Sand (1990), Gerber (1987, 1993), Pelinovsky (2003). The statistics of caustics with application to calculation of the freak wave formation probability was studied in the paper of White and Fornberg of 1998.

On our opinion, a connection between freak wave generation and caustics for swell or wind-driven sea is the indisputable fact. However, this is not the end of the story. Focusing of ocean waves by an inhomogeneous current is a pure linear effect. Meanwhile, no doubts that freak waves are essentially nonlinear objects. They are very steep. In the last stage of their evolution, the steepness becomes infinite, forming a "wall of water". Before this moment, the steepness is higher than one for the limiting Stokes wave. Moreover, a typical freak wave is a single event (see, for instance Divinsky et al 2003). Before breaking it has a crest, three-four (or even more) times higher than the crests of neighbor waves. The freak wave is preceded by a deep trough or "hole in the sea". A characteristic life time of a freak wave is short - ten of wave periods or so. If the wave period is fifteen seconds, this is just few minutes. Freak wave appears almost instantly from a relatively calm sea. Sure, these peculiar features of freak waves cannot be explained by a linear theory. Focusing of ocean waves creates only preconditions for formation of freak waves, which is a strongly nonlinear effect.

It is natural to associate appearance of freak waves with the modulation instability of Stokes waves. This instability is usually called after Benjamin and Feir, however, it was first discovered by Lighthill in 1965. The theory of instability was developed independently by Benjamin and Feir in 1967 and by Zakharov in 1966. Feir in 1967 fist observed the instability experimentally.

Slowly modulated weakly nonlinear Stokes wave is described by nonlinear Shrödinger equation (*NLSE*), derived by Zakharov in 1968. This equation is integrable (see Zakharov, Shabat 1972) and is just the first term in the hierarchy of envelope equations describing packets of surface gravity waves. The second term in this hierarchy was calculated by Dysthe in 1979, the next one was found a few years ago by Trulsen and Dysthe (1996). The Dysthe equation was solved numerically by Ablovitz and his collaborates (2000, 2001).

Since the first work of Smith (1976), many authors tried to explain the freak wave formation in terms of NLSE and its generalizations, like Dysthe equation. A vast scientific literature is devoted to this subject. The list presented below is long but incomplete: Ablovitz et al 2000, 2001; Onorato et al 2001, 2001, 2002; Peregrine 1983; Peregrine et al 1988; Trulsen and Dysthe 1996; Trulsen et al 1997; Trulsen 2000.

One cannot deny some advantages achieved by the use of the envelope equations. Results of many authors agree in one important point: nonlinear development of modulational instability leads to concentration of wave energy in a small spatial region. This is a "hint" about possible formation of freak wave. On the other hand, it is clear that the freak wave phenomenon cannot be explained in terms of envelope equations. Indeed, *NLSE* and its generalizations are derived by expansion in series on powers of parameter $\lambda \; \frac{1}{Lk}$, where k is a wave number, L is a length of modulation. For real freak wave $\lambda \sim 1$ and any "slow modulation expansion" fails. However, the analysis in the framework of the NLS-type equations gives some valuable information about formation of freak waves.

Modulation instability leads to decomposition of initially homogeneous Stokes wave into a system of envelope solitons (more accurately speaking - quasi-solitons (Zakharov, Kuznetsov 1998; Zakharov et al 2004). This state can be called "solitonic turbulence", or, more exactly "quasisolitonic turbulence". In the framework of pure NLSE, solitonic turbulence is "integrable". Solitons are stable , they scatter on each other elastically. Even in this simplest scenario, spatial distribution of wave energy displays essential intermittency. More exact Dysthe equation is not integrable. In this model solitons can merge, this effect increases spatial intermittency and leads to establishing of chaotic intense modulations of energy density. So far this model cannot explain formation of freak waves with $\lambda \sim 1$.

This effect can be explained if the envelope solutions of a certain critical amplitude are unstable, and can collapse. In the framework of 1-D focusing NLSE solitons are stable, thus soliton instability and the collapse must have a certain threshold in amplitude. Instability of a soliton of large amplitude and further collapse could be a proper theoretical explanation of the freak wave origin.

This scenario was observed in numerical experiment on the heuristic one-dimensional Maida-McLaughlin Tabak (MMT) model of one-dimensional wave turbulence (Zakharov et al 2004). At a proper choice of parameters this model mimics gravity waves on the surface of deep water. In the experiments described in the cited paper instability of a moderate amplitude monochromatic wave leads first to creation of a chain of solitons, which merge due to inelastic interaction into one soliton of a large amplitude. This soliton sucks energy from neighbor waves, becomes unstable and collapse up to $\lambda \sim 1$, producing the freak wave. We believe that this mechanism of freak wave formation is universal.

The most direct way to prove validity of the outlined above scenario for freak wave formation is a straight numerical solution of Euler equation, describing potential oscillations of ideal fluid with a free surface in a gravitational field. This solution can be made by the method published in several articles (Dyachenko et al 1996; Zakharov et al 2002; Zakharov 1998). This method is applicable in 1+1 geometry; it includes conformal mapping of fluid bounded by the surface to the lower half-plane together with "optimal" choice of variables, which guarantees well-posedness of the equations (Dyachenko 2004). Earlier, in the paper (Zakharov, Dyachenko, Vasyliev 2002) we studied the nonlinear stage of modulation instability for Stokes waves of steepness $\mu = ka 0.3$ and $\mu = 0.1$.

In the present article we perform similar experiment for waves of steepness $\mu 0.15$. This experiment could be considered as a simulation of a realistic situation. If a typical steepness of the swell is 0.06÷0.07, in caustic area it could easily be two-three times more. In the new experiment, we start with the Stokes waver train, perturbed by a long wave with twenty time less amplitude. We observe development of modulation instability and finally, the explosive formation of the freak wave that is pretty similar to waves observed in natural experiments.

2 Basic equations

Suppose that incompressible fluid covers the domain

$$-\infty < y < \eta(x,t). \tag{1}$$

The flow is potential, hence

$$V = \nabla\phi, \nabla V = 0, \nabla^2\phi = 0. \tag{2}$$

Let $\psi = \phi|_{y=\eta}$ be the potential at the surface and $H=T+U$ be the total energy. The terms

$$T = -\frac{1}{2}\int_{-\infty}^{\infty} \psi\phi_n dx, \tag{3}$$

$$U = \frac{g}{2}\int_{-\infty}^{\infty} \eta^2(x,t) dx, \tag{4}$$

are correspondingly kinetic and potential parts of the energy, g is a gravity acceleration and ϕ_n is a normal velocity at the surface. The variables ψ and η are canonically conjugated; in these variables Euler equation of hydrodynamics reads

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\psi}, \frac{\partial\psi}{\partial t} = -\frac{\delta H}{\delta\eta}. \tag{5}$$

One can perform the conformal transformation to map the domain that is filled with fluid,

$$-\infty<x<\infty,-\infty<y<\eta(x,t),z=x+iy$$

in z-plane to the lower half-plane

$$-\infty<u<-\infty,-\infty<v<0,w=u+iv$$

in w-plane. Now, the shape of surface $\eta(x,t)$ is presented by parametric equations

$$y=y(u,t),\ x=x(u,t),$$

where $x(u,t)$ and $y(u,t)$ are related through Hilbert transformation

$$y=\hat{H}(x(u,t)-u),x(u,t)=u-\hat{H}y(u,t). \tag{6}$$

Here

$$\hat{H}(f(u))=P.V.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(u')du'}{u'-u}.$$

Equations (5) minimize the action,

$$S=\int Ldt, \tag{7}$$

$$L=\int\psi\frac{\partial\eta}{\partial t}dx-H. \tag{8}$$

Lagrangian L can be expressed as follows,

Error! (9)

Here f is the Lagrange multiplier which imposes the relation (6). Minimization of action in conformal variables leads to implicit equations [**Error! Reference source not found.**]

$$y_t x_u - x_t y_u = -\hat{H}\psi_u \psi_t y_u - \psi_u y_t + g y y_u + \hat{H}(\psi_t x_u - \psi_u x_t + g y x_u) = 0. \tag{10}$$

System (10) can be resolved with respect to the time derivatives. Omitting the details, we present only the final result

$$Z_t = iUZ_u, \Phi_t = iU\Phi_u - B + ig(Z - u). \tag{11}$$

Here

$$\Phi = \psi + i\hat{H}\psi$$

is a complex velocity potential, U is a complex transport velocity:

$$U = \hat{P}\left(\frac{-\hat{H}\psi_u}{|z_u|^2}\right) \tag{12}$$

and

$$B = \hat{P}\left(\frac{|\Phi_u|^2}{|z_u|^2}\right) = \hat{P}\left(|\Phi_z|^2\right). \tag{13}$$

In (12) and (13) $\hat{P}$ is the projector operator generating a function that is analytical in a lower half-plane

$$\hat{P}(f) = \frac{1}{2}\left(1 + i\hat{H}\right)f.$$

In equations (11)

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All functions z, Φ, U and B are analytic in the lower half-plane $v<0$.

Equations (11) were derived by Dyachenko, Kuznetsov, Spector and Zakharov and reported on April, 1994 on the conference on Nonlinear Wave Phenomena, Tucson, Arizona. Soon, they were used as a base for numerical simulation by Chalikov and Shenin [**Error!**

Reference source not found.]. We published these equations in 1996 [**Error! Reference source not found.**, **Error! Reference source not found.**] and believe that equations (2.10) were not known until 1994; so we call them *DKSZ*-equations. On the contrary, equations (2.11) are not new. Recently we found that they were derived by Ovsyannikov in 1973 [**Error! Reference source not found.**]. We call them thereafter Ovsyannikov's equations, *OE*.

Note, that equation (2.10) can be used to obtain the Lagrangian description of surface dynamics. Indeed, from (2.10) one can get

$$\Psi=\partial^{-1}\hat{H}(y_t x_u - x_t y_u) \tag{14}$$

Plugging (14) into (2.8) one can express Lagrangian L only in terms of surface elevation. This result was independently obtained by A. Balk in 1995 [**Error! Reference source not found.**]. In 2001, A. Dyachenko [**Error! Reference source not found.**] transformed equations (11) to a simple form, which is convenient both for numerical simulation and analytical study. By introducing of new variables

$$R=\frac{1}{Z_w}, \text{and } V=i\Phi_z=i\frac{\Phi_w}{Z_w} \tag{15}$$

one can transform system (11) into the following one

$$R_t=i(UR_w-RU_w), V_t=i(UV_w-RB_w)+g(R-1). \tag{16}$$

Now complex transport velocity U and B

$$U=\hat{P}(V\bar{R}+\bar{V}R) \quad B=\hat{P}(V\bar{V}). \tag{17}$$

Thereafter, we will call equations (16), (17) Dyachenko equations, *DE*.

Both *DKSZ*-equations (2.10) and *OE* (2.11) have the same constants of motion

$$H=-\int_{-\infty}^{\infty}\Psi\hat{H}\Psi_u du+\frac{g}{2}\int_{-\infty}^{\infty}y^2 x_u dy, \tag{18}$$

the same total mass of fluid

$$M=\int_{-\infty}^{\infty}yx_u du, \tag{19}$$

and the same horizontal momentum

$$P_x = \int_{-\infty}^{\infty} \Psi y_u du. \tag{20}$$

The Dyachenko equations (16), (17) have the same integrals. To express them in terms of R and V, one has to perform the integration

$$z = \int_{-\infty}^{u} \frac{du}{R}, \Phi = -i \int_{-\infty}^{u} \frac{V}{R} du. \tag{21}$$

3 Stokes waves

The Stokes wave is a stationary solution of dynamic equation propagating with a constant velocity c:

$$z = z(u-ct) \Phi = \Phi(u-ct) + \lambda t \tag{1}$$

Any form of dynamic equation can be used for construction of this solution but we consider *DKSZ*-equations to be the most convenient model for this purpose. In complex variables z,ϕ these equations read as

$$\begin{aligned} &z_t + \Phi_u = p^-(\bar{z}_t z_u - z_t \bar{z}_u), \\ &\Phi_t - igz + \Phi_t z_u - z_t \Phi_u - igzz_u = p^-(z_t \bar{\Phi}_u - \bar{\Phi}_t z_u - ig\bar{z}z_u) \end{aligned} \tag{2}$$

For stationary Stokes waves we have

$$\Phi = cz + \lambda t, \Phi_t = -c^2 z_u + \lambda,$$

and system (3.2) is reduced to one equation

$$\lambda(1+z_u) - c^2 z^4 - igz - igzz_u + igP^-(\bar{z}z_u) = 0. \tag{3}$$

The constant λ can be found from the condition $\langle z \rangle = 0$, $\lambda = -ig \langle p^-(\bar{z}z_u) \rangle$. The brackets mean averaging by real axis (3.3). One can look for a solution in the form

$$z = i \sum_{n=1}^{\infty} a_n e^{-inku} \tag{4}$$

From (3.3) one gets

$$\lambda=-gk\sum_{m>1}^{\infty} m a_m^2. \tag{5}$$

It is enough to study the case g=1,k=1; in this case a_n satisfy equations

$$(nc^2-1)a_n=\frac{n}{2}\sum_{m=1}^{n-1} a_m a_{n-m}+\sum_{m=1}^{\infty}(n+m)a_{n+m}a_m-n\sum_{m=1}^{\infty} m a_m^2. \tag{6}$$

One can put:

$$a_n=A\lambda_n, \quad \lambda_1=1, \quad A^2=x,$$

where x is a square amplitude of the first harmonic. The first equation in (3.6) gives value of phase velocity of the Stokes wave:

$$c^2=1+\sum_{m=1}^{\infty}\left((m+1)\lambda_{m+1}\lambda_m-m\lambda_m^2\right)x^m, \tag{7}$$

while other equations make possible to express λ_n, $n\geq 2$, as

$$(nc^2-1)\lambda_n=\frac{n}{2}\sum_{m=1}^{n-1}\lambda_m\lambda_{n-m}+\sum_{m=1}^{\infty}x^m\left((n+m)\lambda_{n+m}\lambda_m-nm\lambda_m^2\right). \tag{8}$$

In this expression, all λ_n are presented by Taylor series on parameter x:

$$\lambda_n=\lambda_n^{(0)}+\lambda_n^{(1)}x+\cdots. \tag{9}$$

It is interesting that the leading terms of these series can be found explicitly from recurrency condition

$$\lambda_n^{(0)}=\frac{n}{2(n-1)}\sum_{m=1}^{n-1}\lambda_m^{(0)}\lambda_{n-m}^{(0)}. \tag{10}$$

In particular,

$$\lambda_2^{(0)}=\lambda, \quad \lambda_3^{(0)}=\frac{3}{2}.$$

Taking into account the first terms in expansion on power of x, one obtains after simple calculations

$$\lambda_2=1+\frac{1}{2}x+\cdots$$

$$c^2=1+x+\frac{7}{2}x^2+\cdots.$$

Then, for the Stokes wave of a moderate amplitude one gets

$$y=A\cos ku+A^2\left(1+\frac{1}{2}(Ak)^2\right)\cos 2ku+\frac{3}{2}A^4\cos 3ku+\cdots$$

$$x=U+A\sin ku+A^2\left(1+\frac{1}{2}(Ak)^2\right)\sin 2ku+\frac{3}{2}A^4\sin 3ku \qquad (11)$$

$$c^2=\frac{g}{k}\left(1+(ka)^2+\frac{7}{2}(ka)^4+\cdots\right) \qquad (12)$$

In (3.11) the surface is presented in parametric form. One can express U as a function on x and find the shape of the Stokes wave in physical coordinates.

We believe that the method outlined above is the most economical way to construct the Stokes waves.

4 Modulational instability of Stokes waves

The Stokes wave is unstable with respect to growing of long-scale modulation. This remarkable fact was first established in 1965 by Lighthill [**Error! Reference source not found.**], who calculated a growth-rate of instability in the limit of long-wave perturbation. As far as Lighthill's growth-rate coefficient was proportional to the wave number of perturbation length, the result was in principle incomplete: somewhere at short scales the instability must be arrested. The complete form of the growth-rate coefficient was found independently in 1966 by Zakharov [**Error! Reference source not found.**], [**Error! Reference source not found.**] and in 1967 by Benjamin and Feir [**Error! Reference source not found.**]. The original Zakharov's work of 1966 was published in Russian journal and it's English translation appeared next year, when the article of Benjamin and Feir was already published; so the instability was called by Benjamin and Feir. Actually, it is not unfair because it was Fair who first observed this instability in experiment.

The presented technique based on the conformal mapping makes possible to study modulational instability in a very compact way. It is convenient to use the Dyachenko equation (16), (17). Let g=1, k=1. To study instability of the Stokes wave, propagating with the velocity c>1, one has to go to the moving reference frame by the following change of variables:

$$y=u-ct, \quad \tau=t, \quad R=1-\frac{iV}{c}+r. \tag{1}$$

Then the Dyachenko equations take the form:

$$\begin{aligned} &\frac{\partial}{\partial t}\left(r-\frac{iV}{c}\right)+cr_u=i(\ddot{u}r'-r\ddot{u}') \\ &\frac{\partial V}{\partial t}=i(VV'-B')-\frac{V}{c}B'-\frac{ig}{c}V+gr+i\ddot{u}V' \\ &\ddot{u}=\bar{p}(\bar{r}V+r\bar{V}) \end{aligned} \tag{2}$$

For the stationary wave we have

$$r=0, \quad \ddot{u}=0, \quad \frac{\partial r}{\partial t}=0, \quad \frac{\partial V}{\partial t}=0,$$

so, for its description we obtain the following equation

$$cV_u=i(VV'-B')-\frac{V}{c}B'-\frac{ig}{c}V.$$

The solution of this equation can be found in the form of Fourier series. Again, we put for simplicity g=1,u=1. Then

$$V=\sum_{n=1}^{\infty}V_n e^{-inu}$$

Coefficients V_n can be found either directly for equation (4.3) or expressed in terms of Fourier coefficients a_n of complex elevation z. Indeed,

$$V=-ic\,\frac{\sum_{n=1}^{\infty} na_n e^{-inu}}{1+\sum_{n=1}^{\infty} na_n e^{-inu}}$$

Thus

$$V_1=-ica_1$$

$$V_2=-ic(2a_2-a_1^2). \qquad (3)$$

Linearization of system (4.2) against the stationary solution leads to equations

$$\frac{\partial}{\partial t}\left(r-i\frac{\delta V}{c}\right)+cr_u=0, \qquad (4)$$

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial u}\right)\delta V=i\frac{\partial}{\partial u}(V_0\delta V-\delta B)-\frac{V_0}{c}\delta B_u-\frac{ig}{c}\delta V+gr.$$

System (4.6) contains all information about stability of the Stokes wave.

The modulational instability is described by a perturbation presented as a sum of following harmonics:

$$\delta V,r\equiv e^{-i\kappa u},e^{(1\pm\kappa)u-in\kappa}, \quad n=1,\ldots, \quad \kappa<1.$$

In the leading order of nonlinearity one can put

$$r=p_1e^{-i(1+\kappa)u}+p_2e^{-i(1-\kappa)u},$$

$$V=q_1e^{-i(1+\kappa)u}+q_2e^{-i(1-\kappa)u}. \qquad (5)$$

Plugging (4.7) to (4.6) one obtains closed equations to p_1,p_2,q_1,q_2:

$$\dot{p}_1-\frac{i}{c}\dot{q}_1=ic(1+\kappa)p_1,$$

$$\dot{\bar{p}}_2+\frac{i}{c}\dot{\bar{q}}_2=-ic(1-\kappa)p_2, \qquad (6)$$

$$\dot{q}_1-i\left(\frac{1}{c}-c(1+\kappa)\right)q_1-p_1=V_2(1+\kappa)\bar{q}_2,$$

$$\dot{\bar{q}}_2-i\left(\frac{1}{c}-c(1-\kappa)\right)\bar{q}_2-\bar{p}_2=\bar{V}_2(1-\kappa)q_1.$$

Here V_2 is the amplitude of second harmonics. Assuming $p_1,q_1,\bar{p}_2,\bar{q}_2 e^{i(\Omega+\kappa c)t}$, one gets the following equation for Ω:

$$\left[(\Omega-c)^2-1-\kappa\right]\left[(\Omega+c)^2-1+\kappa\right]=(c^2-\Omega^2)\left(\frac{1}{c}-c\right)^2(1-\kappa^2)$$

To obtain this equation we put

$$|V_2|^2=\left(\frac{1}{c}-c\right)^2.$$

This condition appears from the natural physical requirement: if $\kappa=0$, then $\Omega=0$ is a solution of (4.9). In our approximation $V_2=-ica_1^4, c^2=1+a_1^2$, and condition (4.10) is satisfied. Assuming that $\Omega<<1$, one can simplify (4.9) to quadratic equation, that is equivalent to the equation derived by Zakharov in 1967 [**Error! Reference source not found.**]:

$$\left(\Omega-c+\sqrt{1+\kappa}\right)\left(\Omega-+c-\sqrt{1-\kappa}\right)=(c-1)^2(1-\kappa^2).$$

Assuming $a_1=A$, we put

$$c=1+\frac{1}{2}A^2,\ \sqrt{1+\kappa}\,1+\frac{1}{2}\kappa-\frac{1}{8}\kappa^2,\ \sqrt{1-\kappa}\,1-\frac{1}{2}\kappa-\frac{1}{8}\kappa^2,$$

and get from (4.11) the standard growth-rate of modulational instability

$$\Omega^2=\frac{1}{8}\left(A^2\kappa^2+\frac{1}{8}\kappa^4\right),$$

the result that was obtained by Zakharov in 1967, 1968 [**Error! Reference source not found.**, **Error! Reference source not found.**], and by Benjamin and Feir in 1967 [**Error! Reference source not

found.]. Lighthill in 1965 found long-wave asymptotic of the instability growth-rate,

$$\Omega^2 = -\frac{1}{8} A^2 \kappa^2,$$

with the maximum value of the growth-rate,

$$\Omega^2 = -\frac{1}{4} A^2 \kappa^2,$$

achieved at

$$\kappa^2 = 4A^2.$$

The technique developed above makes possible to study the modulational and other instabilities with any arbitrary accuracy.

5 Envelope equations

The free surface equation written in conformal variables is a perfect starting point for derivation of envelope equation. We will start with system (3.2) and decompose its solution as follow:

$$z = z_0 + z_1 + z_2 + \cdots, \quad \Phi = \Phi_0 + \Phi_1 + \Phi_2 + \cdots,$$

where z_n, Φ_n are close to

$$e^{in\omega t - in\kappa u}, \quad \omega\sqrt{gk}.$$

We will assume that the solution is weakly nonlinear, and all components in (5.1) are small. They are of different orders of magnitude. Let the leading terms z_1, Φ_1 be proportional to [epsilon]<<1. Then one has to separate fast and slow evolutions in time and space by introducing of "slow" variables y, τ:

$$\frac{\partial}{\partial t} = \left(i\omega n + \text{[epsilon]} \frac{\partial}{\partial \tau} \right) z_n, \quad \frac{\partial}{\partial u} = \left(ikn + \text{[epsilon]} \frac{\partial}{\partial y} \right) z_n.$$

We can do the same for Φ_n, if $n>0$. For $n=0$, we have to put

$$\frac{\partial}{\partial t}\Phi_0 = \left(\lambda + [\text{epsilon}]\,\frac{\partial}{\partial \tau}\right)\Phi_0,$$

where λ is some constant. After performing decomposition (5.1), system (3.2) turns to an infinite system of coupled equations for any n.For n=0, in the leading order one gets

$$z_\tau + \Phi_y = 0, \quad \lambda - igz_0 = -gkP^-(|z_1|^2).$$

Assuming that $<z_0>=0$, one obtains

$$\lambda = -gkP^- <|z_1|^2> = -gk<|z_1|^2>,$$

$$z_0 = -ikP^- \left(|z_1|^2 - <|z_1|^2>\right).$$

For the imaginary part of (5.6) one gets the equation

$$y = -\frac{1}{2}k\left(|z_1|^2 - <|z_1|^2>\right)$$

that describes deformation of the sea level due to the presence of a quasimonochromatic wave. For wave packets of small amplitude we can put

$$\frac{\partial}{\partial \tau} V_{group} \frac{\partial}{\partial y} \frac{\omega}{2k} \frac{\partial}{\partial y},$$

hence

$$\Phi_0 \frac{i\omega}{k} z_0.$$

For the second harmonics we have the system

$$z_{2t} + \Phi_{2u} = 0$$
$$\Phi_{2t} - igz_2 = igz_1 z_{1u}. \quad (1)$$

Using (5.2), we get in the leading order

$$z_2=ikz_1^2, \quad \Phi_2=-i\omega z_1^2.$$

We will denote thereafter $z_1=z$. Then, plugging z_0, z_2 to equations for z_1, we receive after simple calculations the following equation:

$$ig\frac{\partial z}{\partial u}+\frac{\partial^2 z}{\partial t^2}+gk^3|z|^2z=0. \tag{2}$$

This equation is the Nonlinear Schrodinger equation. Meanwhile, it differs from the "standard" NSLE derived by Zakharov in 1968. This equation is written in "natural" variables u, t and is just a first approximation. After some calculations, which we omit, one can derive a more accurate equation:

$$ig\frac{\partial z}{\partial u}+\frac{\partial^2 z}{\partial \tau^2}+gk^3|z|^2z=\omega k^2\left(z\frac{\partial}{\partial \tau}\hat{H}|z|^2-i\frac{\partial}{\partial \tau}|z|^2z\right). \tag{3}$$

Here

$$\tau=t-\frac{uk}{\omega}$$

and

$$\hat{H}z=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{z(\tau')}{\tau'-\tau}d\tau'$$

is the Hilbert transform in the time domain. Equation (5.10) is equivalent to the Dysthe equation [**Error! Reference source not found.**], but plays a role of time, when τ plays a role of spatial coordinate. Such equations are common in nonlinear optics. This is a Hamiltonian system

$$ig\frac{\partial z}{\partial u}=\frac{\delta H}{\delta \bar{u}} \quad (4)$$

$$H=\int_{-\infty}^{\infty}\left[|z|^2-\frac{1}{2}qk^2|z|^4+\frac{1}{2}\omega k^2|z|^2\hat{H}\frac{\partial}{\partial\tau}|z|^2-\frac{i\omega k^2}{4}|z|^2(z_\tau\bar{z}-\bar{z}_\tau z)\right]d\tau$$

Introduced variables u,τ are more convenient than traditional envelope variables, because in the new variable the higher-order derivatives by τ do not appear.

Equation (5.9) is an integrable system (Zakharov, Shabat 1972). More accurate equation (5.10) is certainly non-integrable. This conclusion implicitly follows from numerical experiments by Ablovitz et al ([**Error! Reference source not found.**]). These authors solved the Dysthe equation equivalent to (5.10) and observed onset of chaos. The *NSLE* (5.9) has solitonic solutions:

$$z=\frac{e^{i\lambda^2 u}\sqrt{2}\lambda}{k\cosh\lambda\omega t},\quad \omega^2=\sqrt{gk}.$$

These solutions are stable. They interact elastically and cannot merge.

Equation (5.10) also has solitonic solution

$$z=e^{i\lambda^2 u}\phi(\tau,\lambda),$$

where ϕ satisfies the equation

$$-\lambda^2 g\phi+\frac{\partial^2\phi}{\partial\tau^2}+gk^3|\phi|^2\phi=\omega k^2\left(\phi\frac{\partial}{\partial\tau}H|\phi|^2-i|\phi|^2\frac{\partial\phi}{\partial\tau}\right)$$

The solution of this equation is a complex function; this function tends to *NSLE* soliton (5.12) at $\lambda\to 0$. As far as equation (5.10) is not integrable, one can expect that solitons in this equation can merge. The most important question that remains open is the stability of these solitons.

6 Numerical approach

Many numerical schemes were developed for the solution of Euler equations describing a potential flow of free-surface fluid in a gravity field. Most of them use the integral equations, which solve the boundary-value problem for Laplace equation [**Error! Reference source not found.**], [**Error! Reference source not found.**], [**Error! Reference source not found.**]. A survey of the method can be found in [**Error! Reference source not found.**].

In this article we apply the spectral code to solve the Dyachenko equations formulated in Chapter 2. We should mention that conformal mapping is a routine approach to study the stationary Stokes wave. The equations for their Fourier coefficients, equivalent to system (3.2), were solved numerically by many authors (see, for instance [**Error! Reference source not found.**]). The idea to implement conformal mapping for simulation of essentially nonstationary wave dynamics emerged in the beginning of eighties (see Meison, Orzag 1981). As far as close *DKSZ* and *DE* equations were not derived at that time, and *OE* equation was not known to most of hydrodynamists, the authors used the quasi-Lagrangian approach to fluid dynamics.After some experiments and discussion of their results, the idea to use the conformal mapping war abundant for the following reason: conformal mapping is not good for resolution of wedge-type singularities, naturally appearing on the free surface of fluid. This reason is serious if the spatial mesh is sparse. However, modern computers make possible to use very fine meshes consisting of more than million points or spectral modes. Thus, this argument is not tenant any more.

The *DKSZ*-equations, not resolved with respect to time derivatives, are not convenient for numerical simulation. As soon as explicit *OE*-equations were rediscovered in 1993, two groups of researches started to develop numerical codes for their solution. The works of Chalikov and Sheinin were summarized in their paper of 1998 (see also [**Error! Reference source not found.**]). First results, obtained by our group, were published in 1996, [**Error! Reference source not found.**]. However, since 2001 we switched on solution of *DE*'s. We have very serious reasons for this choice.

Of course, *OE* and *DE* are quite equivalent from the physical view point. Their solution on computer consumes comparable time. However, it was established empirically that numerical solution of *DE* is much more stable, robust, and reliable procedure then solution of *OE*, performed with the same method. So far, we don't understand a reason for this difference. It might corroborate with a following remarkable mathematical fact: the "naive" WKB-type test on well-posedness of the initial value problem leads to positive results for *DE* and negative for *OE*-equations (see Application). This fact was discovered by Dyachenko, [**Error! Reference source not found.**]. The numerical code for solution of *DE*'s was developed by Zakharov, Dyachenko and Vasyliev (see [**Error! Reference source not found.**]).

Dyachenko equations are not *PDE*'s. They are "Hilbert-differential equations, which include together with derivatives with respect to u,

$$f \to \frac{\partial}{\partial u},$$

the operation of making the Hilbert transformation

$$f \to Hf.$$

From analytic view-point these two operations are completely different. But from numerical view-point they are similar. Indeed, in terms of Fourier transform operation (2.14) means

$$f_k \to ikf_k,$$

while operation (6.2) means

$$f_k \to isignf_k$$

From computational view-point both operations are of same level of difficulty. Moreover, Dyachenko equations have polynomial (actually quadratic) nonlinearity. Thus they are ideal objects for the implementation of the the standard spectral code. As far as all unknown functions are analytic in the lower half-plane, all negative harmonics are zero. A number of positive harmonics was different in different times; the maximum number was two millions. We use the fourth order Runge-Kutta integration scheme to solve the equation in the domain of length 2π.

7 Freak waves as a result of modulational instability

In our previous paper [**Error! Reference source not found.**] we solve the Dyachenko equation in periodic box of length 2π. We put the gravity acceleration g=10 and choose the initial data as a superposition of monochromatic wave with wave number k=-50 and Gaussian random noise:

Error!

The initial spectra of function R is shown on Figure 1 (it mark is t=0), and a part of initial surface $y(x)$ is shown on Figure 2.

Forcing and dumping are absent in these computations. The total energy remains constant during simulation up to t=80. After that spectra some wider that it needs to increase the number of Fourier modes, or to stop the simulation.

The initial spectra (t=0) and its evolution t=40 and t=80 are shown on Figure 1.

Figure 1: Development of modulational instability: spectra $R(k)$ of function $R(u)$ at different moments of time t.

We can see, how smooth continuous spectra develop from single spectral harmonic with a bit of noise. The spectral "tail" steady propagates in the area of high wave numbers. This process can be interpreted as formation of singularities on the crests of individual waves, another words, as an onset of wave-breaking. To continue calculations beyond the moment t=80 one should include into calculation the dissipation.

The maximal value of growth-rate of modulational instability,

$$\gamma_{max} = \omega \cdot \frac{1}{2}(ka)^2,$$

is reached for perturbations with wave number p,

$$p 2(ka) \cdot k.$$

In our computations:

$$\mu = ka 0.1 \quad \omega = \sqrt{500} 24,$$

$$\gamma_{max} 0.12, \quad \gamma_{max}^{-1} 8, \quad p 5k. \qquad (1)$$

So, the total process takes about ten inverse growth rates. During the most part of this time we observed linear exponential growth of initial perturbation. On the last stage of the process, we see fast formation of freak waves with an amplitude that is more than three times exceeding the initial one. The distance between freak waves is 5-7 periods of the initial wave. Comparing with (7.4), we conclude that the distance between freak waves is close to the length of the most fast growing modulation.

Figure 2: Development of freak waves due to nonlinear interaction: the surface of fluid at initial time t=0.

Figure 3: Development of freak waves due to nonlinear interaction: the surface of fluid at t=80.

Figure 2 presents the initial shape of fluid surface. One can see that the initial random noise imposed on the monochromatic wave is very small. Figure 3 presents the shape of the surface at the end of our calculation. One can see the formation of "freak" wave with the amplitude more than three times exceeding the initial level. Figures 4, 5, 6 display distribution of densities of kinetic, potential and total energy; they are even more spectacular. One can see that the total energy density in freak waves exceeds the average level almost in two orders of magnitude. In these experiments the total number of harmonics N=12,228 was relatively small, thus the accuracy is limited.

Figure 4: Development of freak waves due to nonlinear interaction: density of kinetic energy $\omega_T(x)$ at t=80.

Figure 5: Development of freak waves due to nonlinear interaction: density of potential energy $\omega_U(x)$ at t=80.

Figure 6: Development of freak waves due to nonlinear interaction: density of potential energy $\omega(x)$ at t=80.

Our recent experiments are much more accurate: we use 10^5 to $2 \cdot 10^6$ harmonics. We solve Dyachenko equations in the periodic domain of length 2π, putting g=1. The initial data are chosen as a combination of the exact Stokes wave (wave number k=10, steepness ka=μ=0.15) and a long monochromatic wave with wave number k=1 and a moderate amplitude $5 \cdot 10^{-2}$. This relatively high level of perturbation is chosen deliberately to make shorter the period of exponential instability growth that is not interesting for us. At given conditions, the maximum growth-rate is

$$\gamma_{max} \frac{\sqrt{10}}{2} \cdot 0.15^2 0.035$$

and γ^{-1}_{max}28.6. The period of initial wave $T_0 = \frac{2\pi}{\sqrt{3}}$. The simulation is continued until T458.842, that is more than sixteen inverse growth-rates. We performed the computations with double precision, with the number of modes doubled as far as amplitude of the last mode reached 10^{-15}. The maximum number of modes was two millions.

We observed a short period of exponential growth of perturbation, then, some intermediate regime of intensive modulation, which ends up with explosive formation of one single freak wave. Pictures of surface shape between T=442 and T=456,56 are presented on Figures 7 8, 9 10, 11.

Figure 7: The shape of surface at T=422.

Figure 8: The shape of surface at T=446.

Figure 9: The shape of surface at T=455.25.

Figure 10: The shape of surface at T=458.03.

Figure 11: The shape of surface at T=458.56.

The time interval T=442 and T=456,56 contains seven periods of initial wave only. One can see fast, non-monotonic formation of the freak wave. At this moment the freak wave is more steep than the Stokes wave of critical amplitude. Amplitudes of waves, preceding the freak wave, are relatively small. One can see a trough just ahead of freak wave. Figure 12 demonstrates the fine structure of surface shape near the wave crest.

Figure 12: The shape of surface near the wave crest at T=458.61.

We managed to continue our simulation until the moment T=458.842. The zoomed shape of the surface at that time is presented on Figure 13.

Figure 13: The shape of surface near the wave crest at T=458.842.

One can see that near the crest the front face of the wave is very steep. This is really "wall of water". In some region the steepness is even negative. The curvature of the shape is plotted on Figure 14.

Figure 14: Curvature of the surface at T=458.842.

This is actually a breaking wave. Note, that the maximum value of the freak wave height is three times higher than the height of the initial wave. Growing of wave hight up to this level from the level of significant wave height takes less than ten wave periods. This is a really fast process; it is three times faster than the developing of modulational instability.

Figure 15 display the evolution of spatial density of kinetic energy in the domain [5.5-9.5], where the breaking takes place.

Figure 15: . The density of kinetic energy at the moments of time: a) T=443, b) T=448, c) T=453, d) T=455, e) T=456, f) T=458.5

One can see that this evolution is non-monotonous. The density oscillates in time and finally condensates in one very narrow wave crest. In general, the whole process of freak wave formation is non-monotonous. We can say that the freak wave "runs" over wave crests until one of them reaches extremely high amplitude. This behavior can be easily explained by difference of phase and group velocities: the energy propagates with group velocity that is twice less than the phase velocity. Figure 16 demonstrates distribution of horizontal momentum before and after breaking, at T=455 and T=456. One can see that the process of momentum concentration in a moving but localized area is monotonous. Definitely, this behavior can be explained by the fact that momentum is a conservative quantity.

Figure 16: . Distribution of momentum before and after breaking

8 Do freak waves appear from quasisolitonic turbulence?

Let us summarize the results of our numerical experiments. Certainly, they reproduce the most apparent features of freak waves: single wave crests of very high amplitude, exceeding the significant wave height more than three times, appear from "nowhere" and reach full height in a very short time, less than ten periods of surrounding waves. The singular freak wave is proceeded by the area of diminished wave amplitudes. Nevertheless, the central question about the physical mechanism of freak waves origin is still open.

In our experiments, the freak wave appears as a result of development of modulational instability, and it takes a long time for the onset of instability to create a freak wave. Indeed, the level of perturbation in our last experiment is relatively high. The two-three inverse growth-rate is enough to reach the state of full-developed instability, when the initial Stokes wave is completely decomposed. Meanwhile, the freak wave appears only after fifteenth inverse growth-rates of instability. What happens after developing of instability but before formation of freak wave?

During this relatively long period of time, the state of fluid surface can be characterized as quasisolitonic turbulence, that consists of randomly located quasisolitons of different amplitudes moving with different group velocities. Such quasisolitonic turbulence is studied in the recent work of Zakharov, Dias and Pushkarev ([**Error! Reference source not found.**]) in a framework of so-called defocusing *MMT* model.

$$i\frac{\partial\Psi}{\partial t}=\left|\frac{\partial}{\partial x}\right|^{1/2}\Psi+\left|\frac{\partial}{\partial x}\right|^{3/4}\left(\left|\left|\frac{\partial}{\partial x}\right|^{3/4}\Psi\right|^{2}\left|\frac{\partial}{\partial x}\right|^{3/4}\Psi\right) \qquad (1)$$

This is a heuristic model description of gravity surface waves in deep water. In this model, quasisolitons of small amplitude are stable, interact inelastically and can merge. Above some critical level quasisolitons of large amplitude are unstable. They collapse in finite time forming very short wave pulses, which can be considered as models of freak waves. Equation (8.1) has the exact solution:

$$\Psi=A\,e^{ikx-i\omega t}$$

$$\omega=k^{1/2}\left(1+k^{5/2}A^{2}\right). \qquad (2)$$

This solution can be constructed as a model of the Stokes wave and is unstable with respect to modulational instability. Development of this instability was studied numerically. On the first stage, the unstable monochromatic wave decomposes to a system of almost equal quasisolitons. Then, the quasisolitonic turbulence is formed: quasisolitons move chaotically, interact with each other, and merge. Finally they create one large quasisoliton, which exceeds threshold of instability and collapses, creating a freak wave.

One can think that a similar scenario of freak wave formation is realized in a real sea. We like to stress that the key point in this scenario is the quasisolitonic turbulence and not the Stokes wave. The Stokes wave is just a "generator" of this turbulence. The quasisolitonic turbulence can appear as a result of instability of narrow spectral distributions of gravity waves.

The formulated above concept is so far a hypothesis, which has to be confirmed by future numerical experiments.

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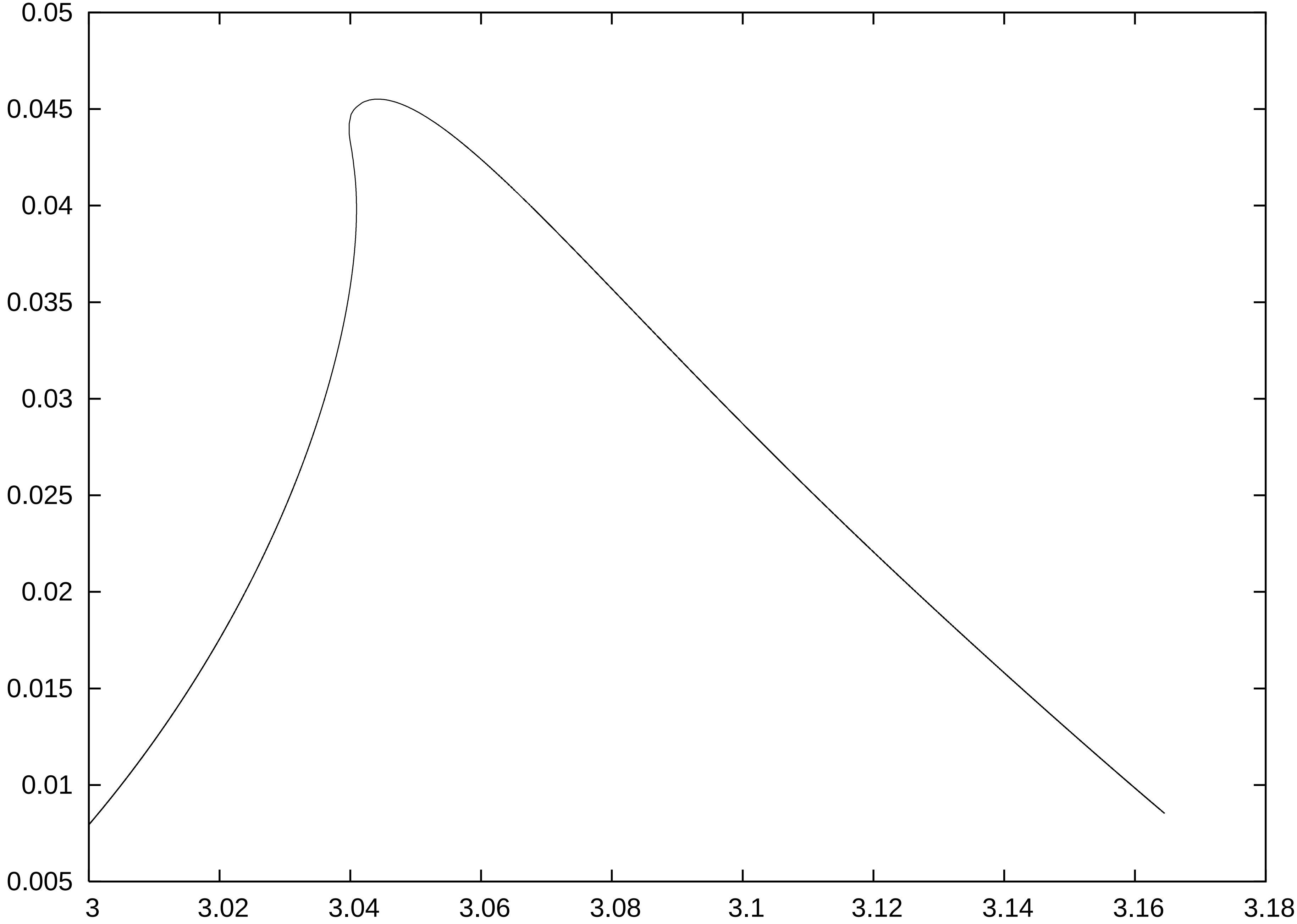
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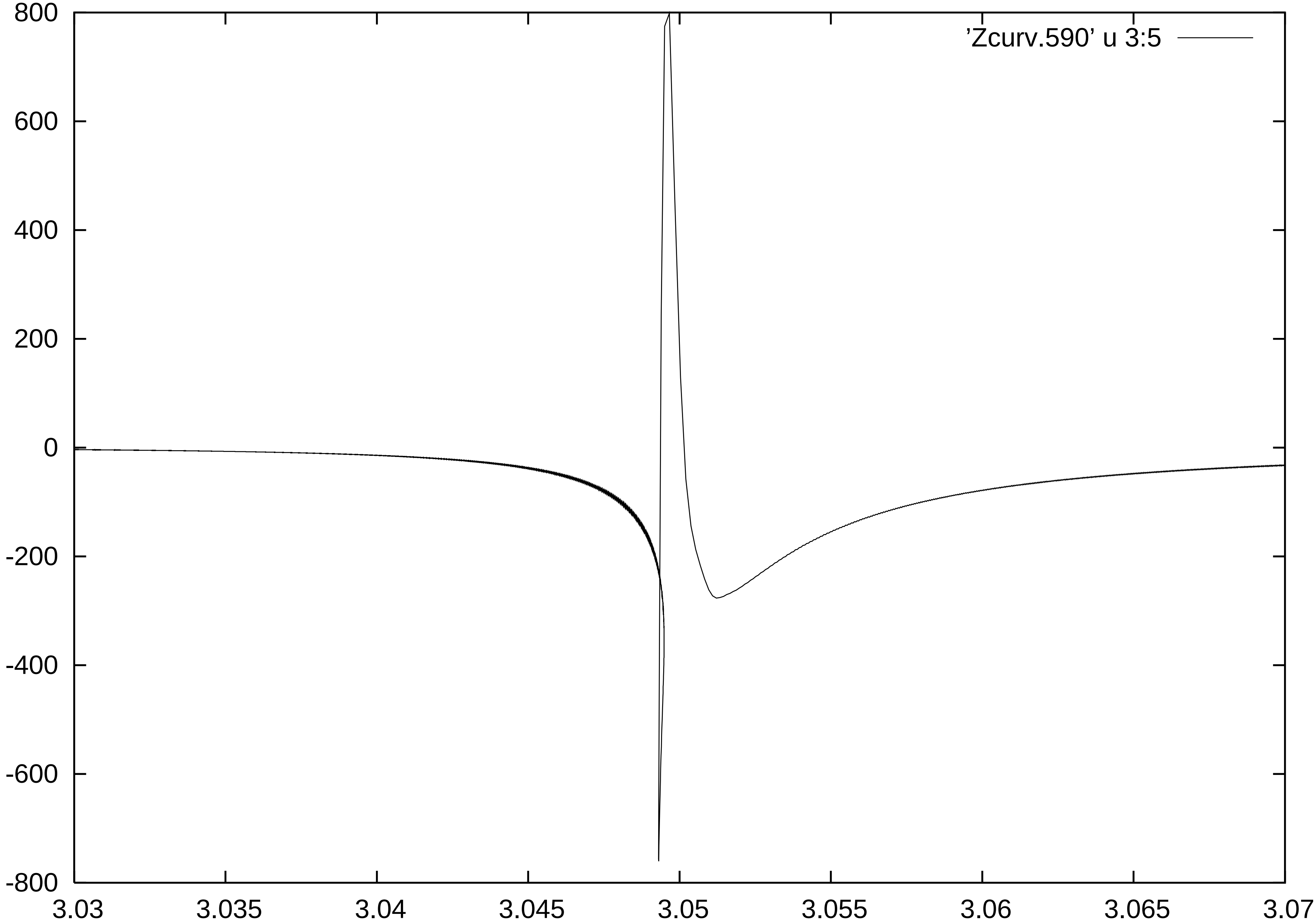

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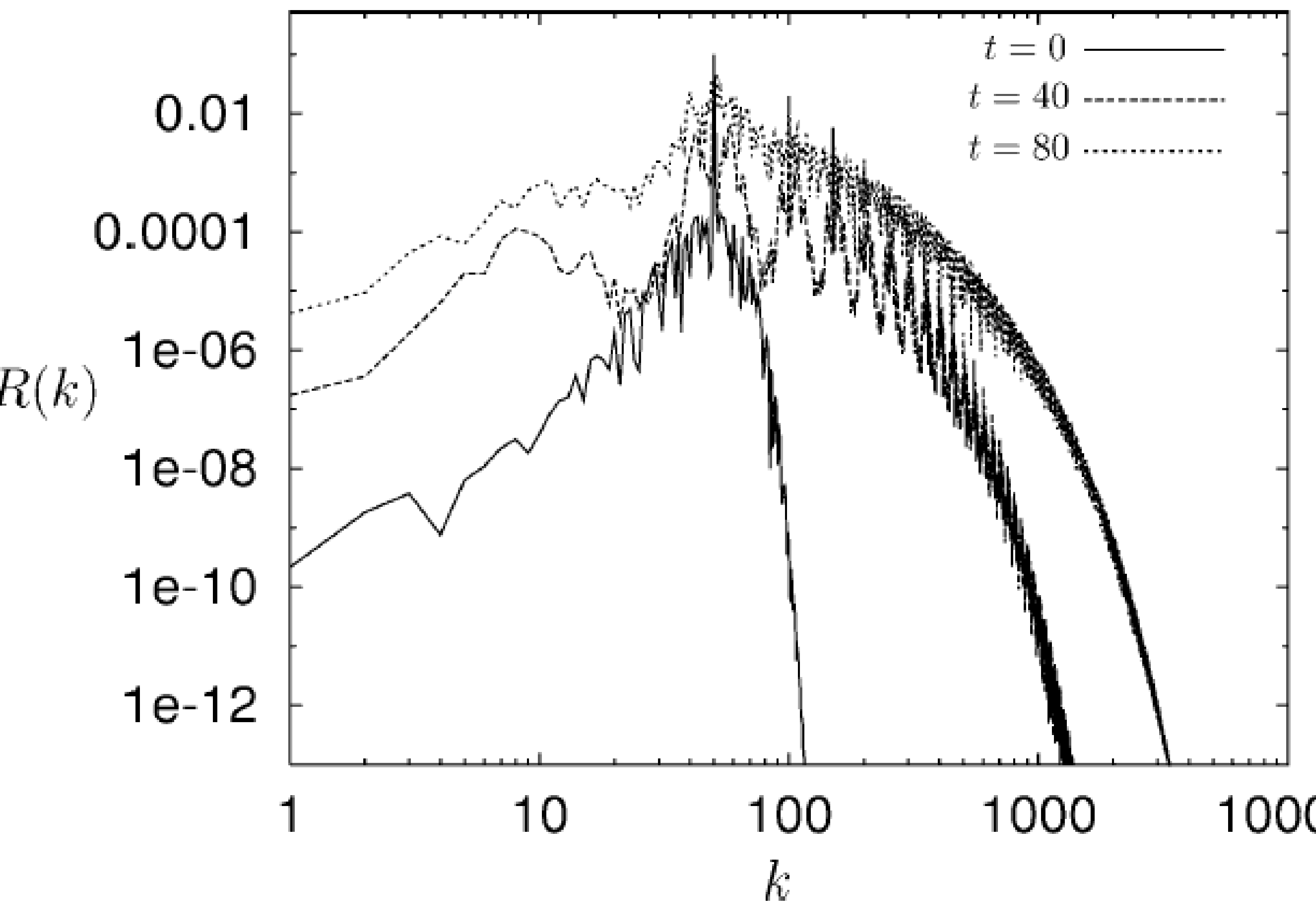

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0.0001
1e-06
1e-08
1e-10
1e-12
R(k)
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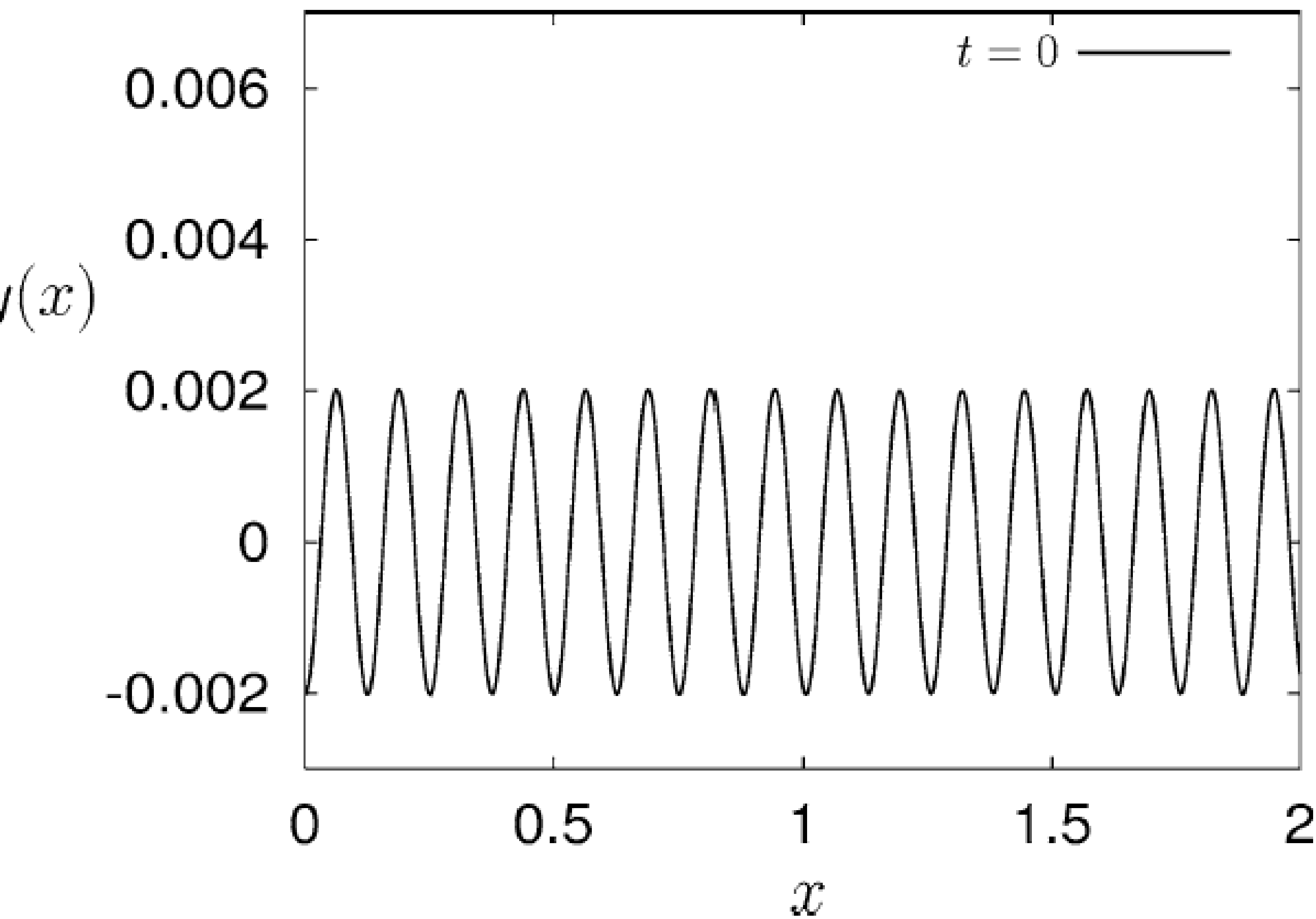
10
100
1000
k



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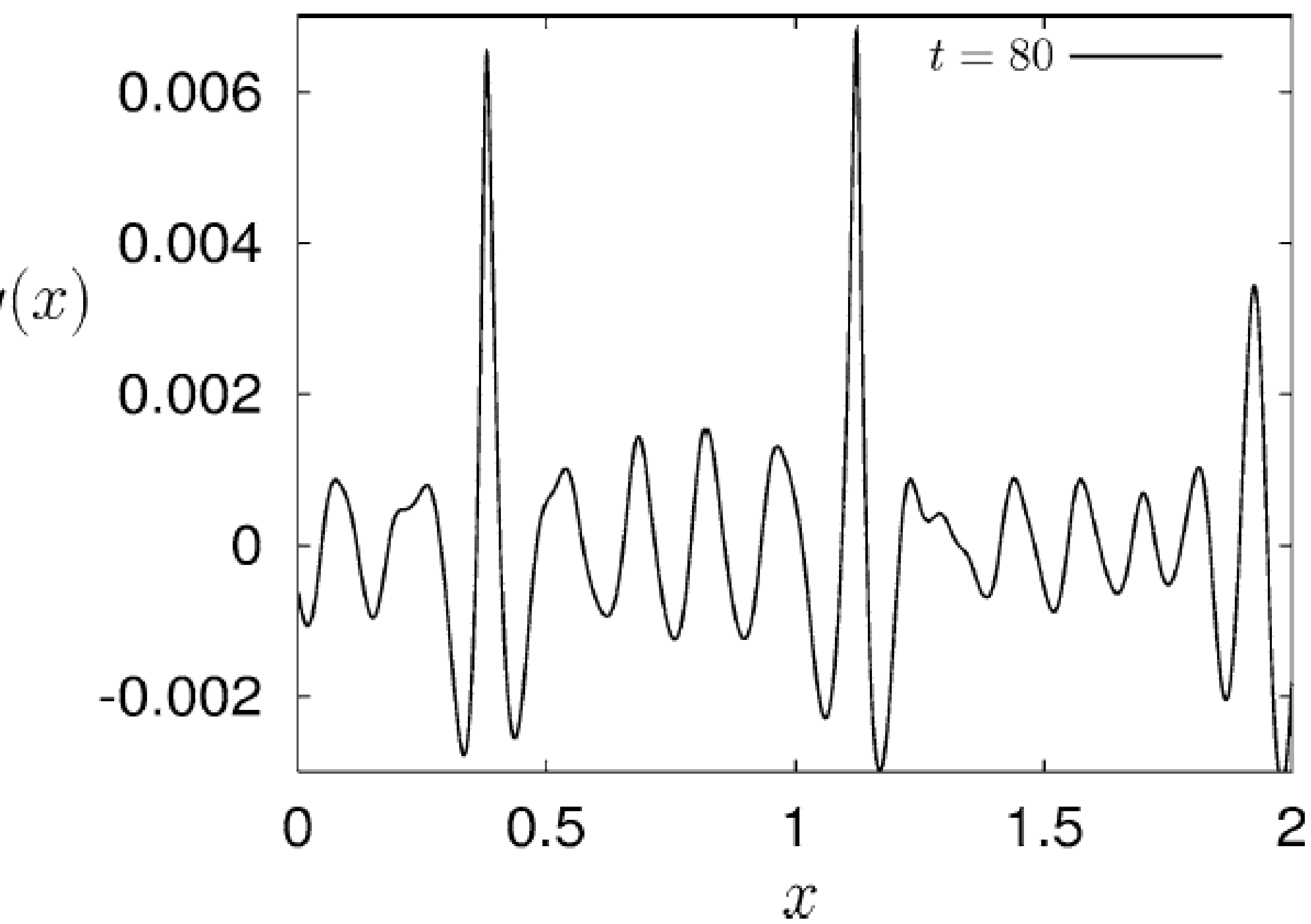
0.004
0.002
0
-0.002
(x)
0
0.5
1
1.5
2
x



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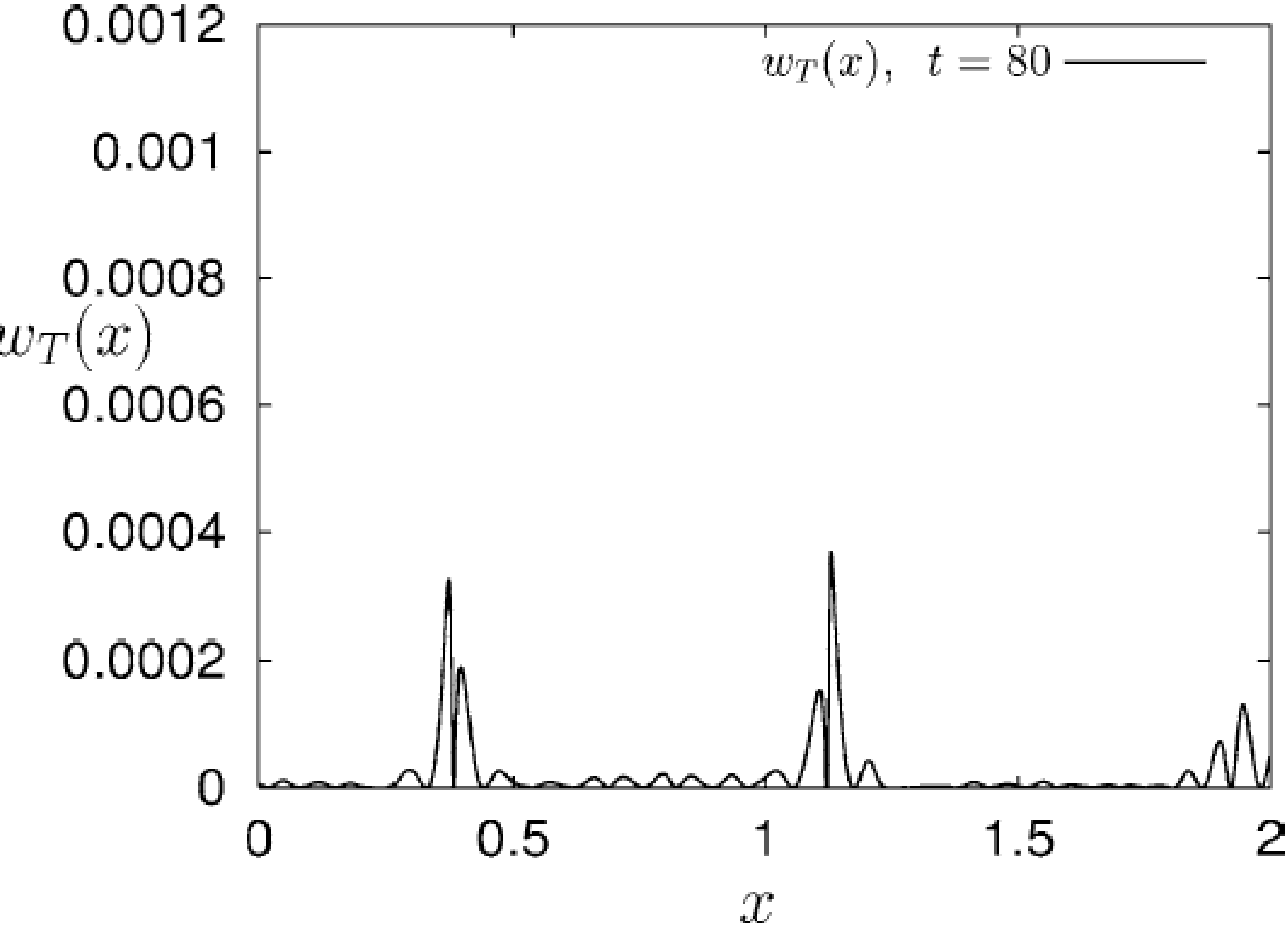
0.002
0
-0.002
0
0.5
1
1.5
2
x



$w_T(x), \ t = 80$
$w_T(x)$
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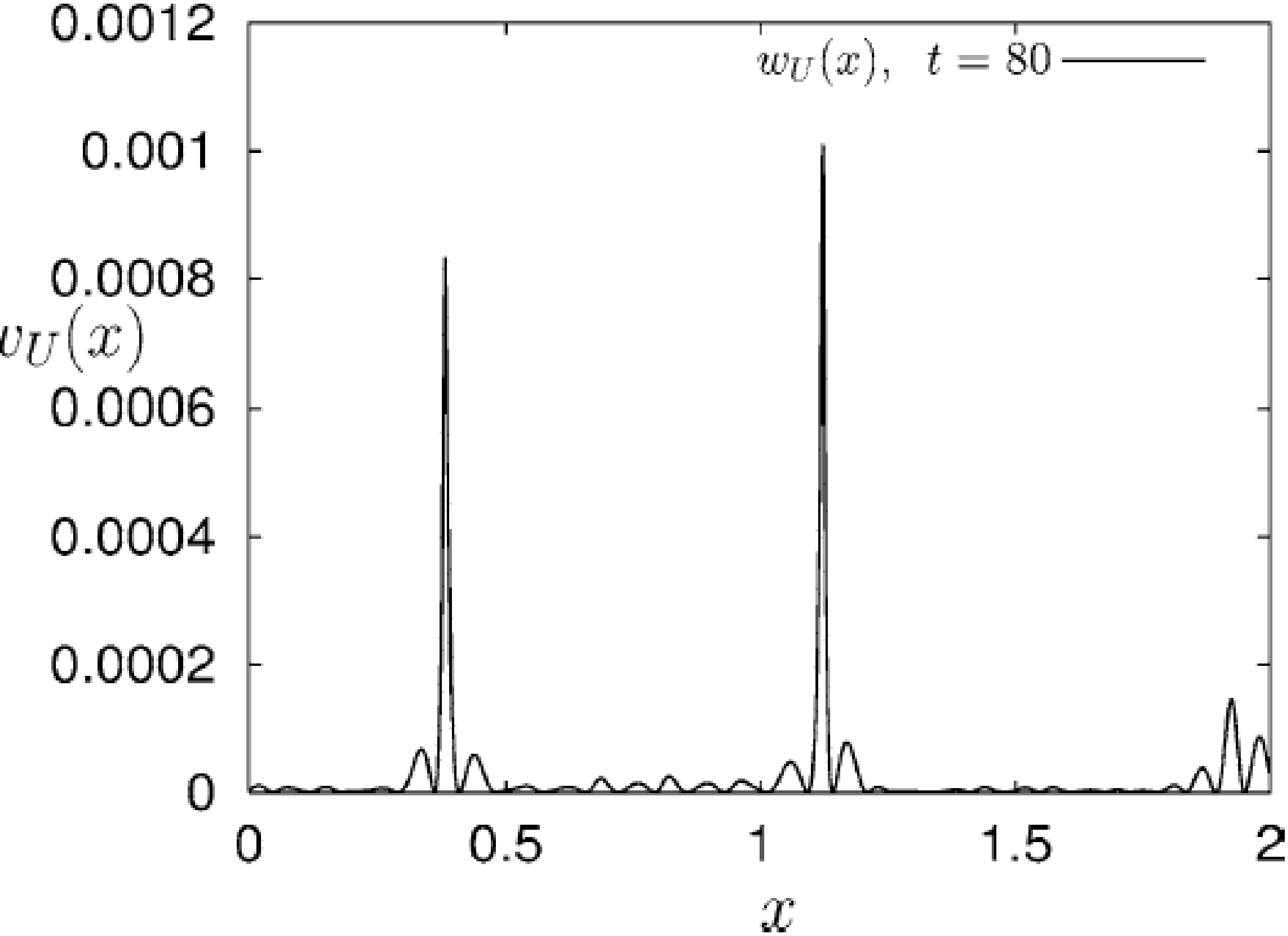
0.0006
0.0004
0.0002
0
0
0.5
1
1.5
2



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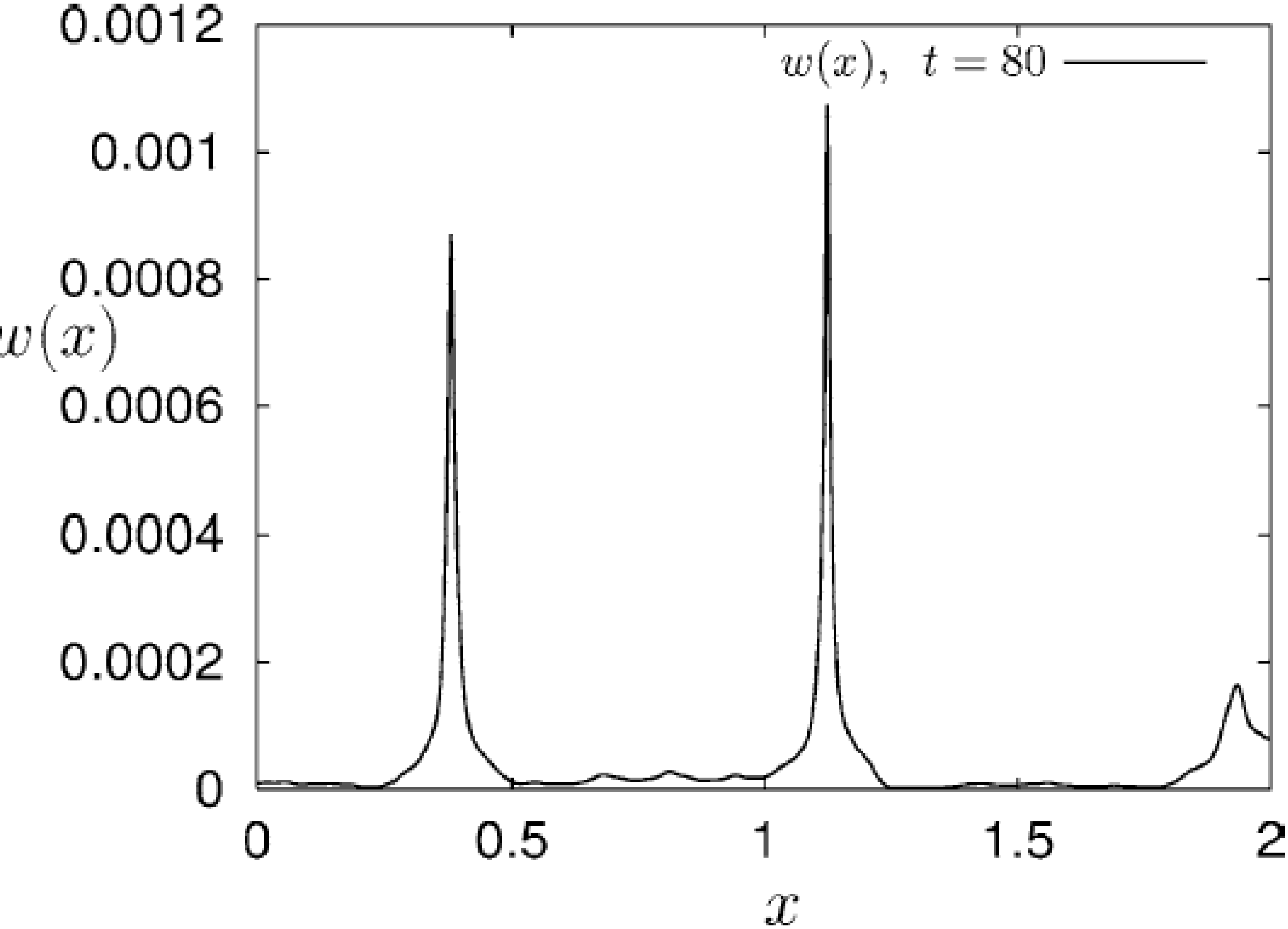
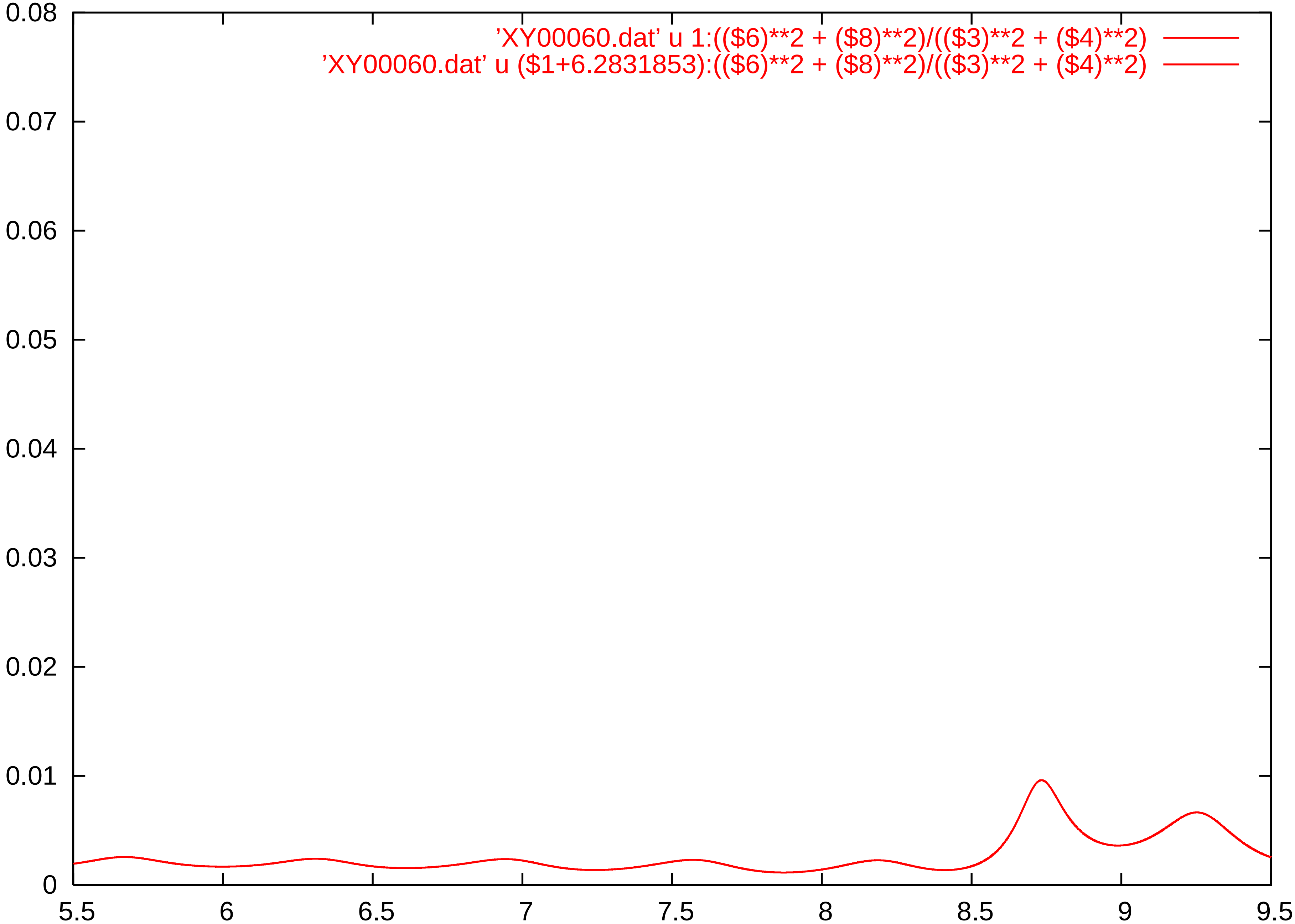
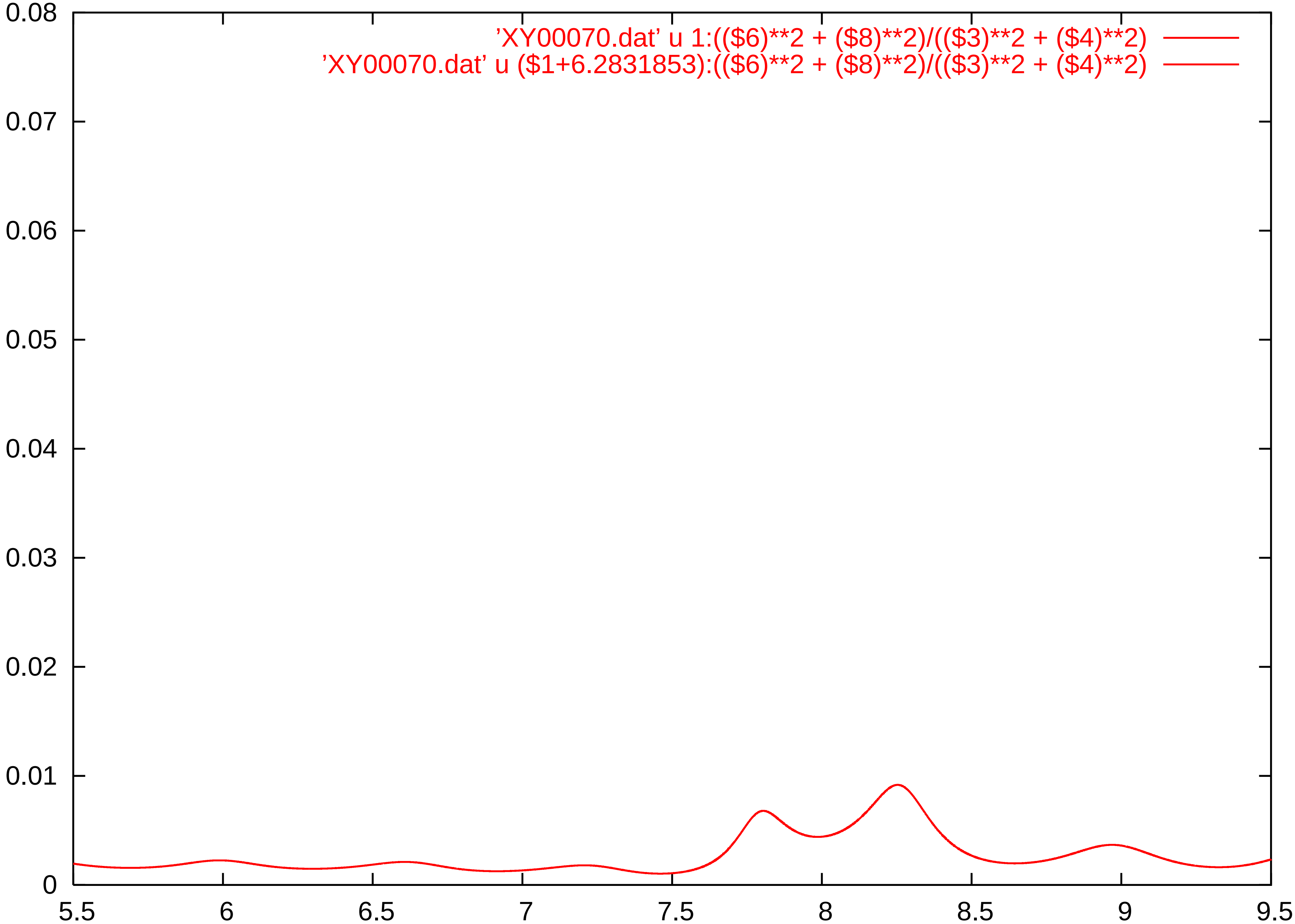
0.0004
0.0002
0
$w_U(x)$
$w_U(x), \quad t = 80$
0
0.5
1
1.5
2
x

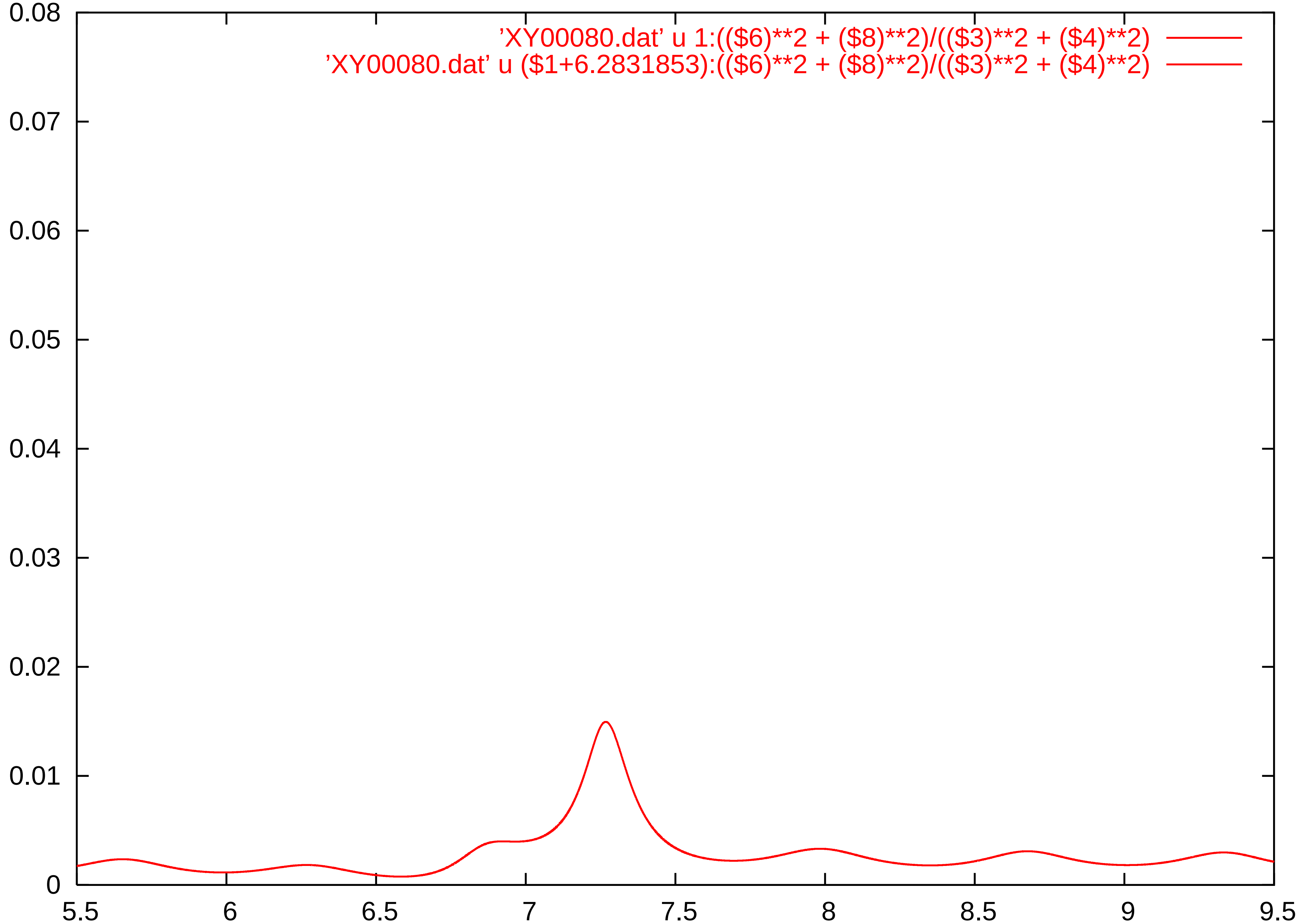


0.0012
0.001
0.0008
0.0006
0.0004
0.0002
0
$w(x)$
0
0.5
1
1.5
2
x
$w(x), \ t = 80$

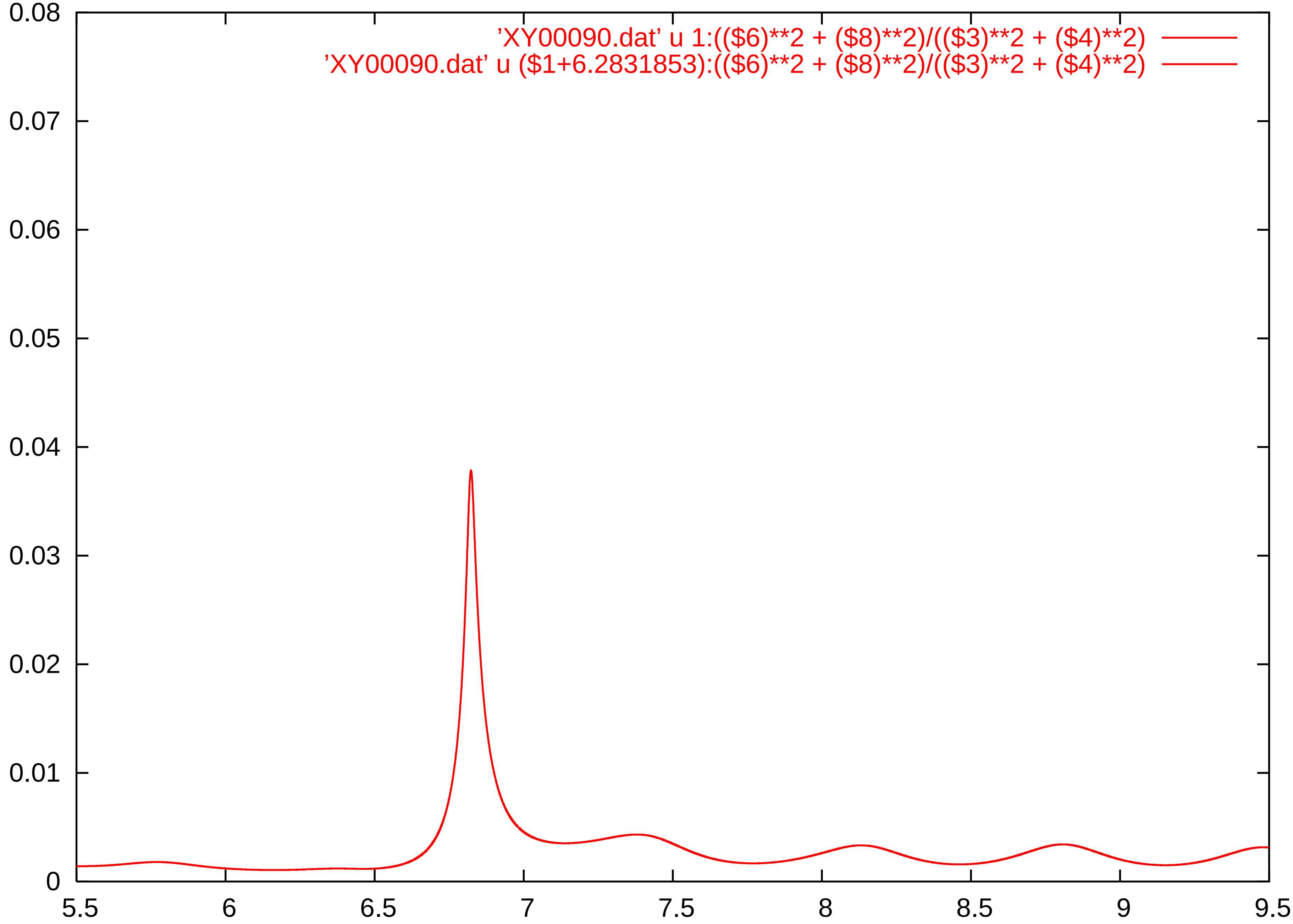


'XY00060.dat' u 1:(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
'XY00060.dat' u ($1+6.2831853):(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
5.5
6
6.5
7
7.5
8
8.5
9
9.5

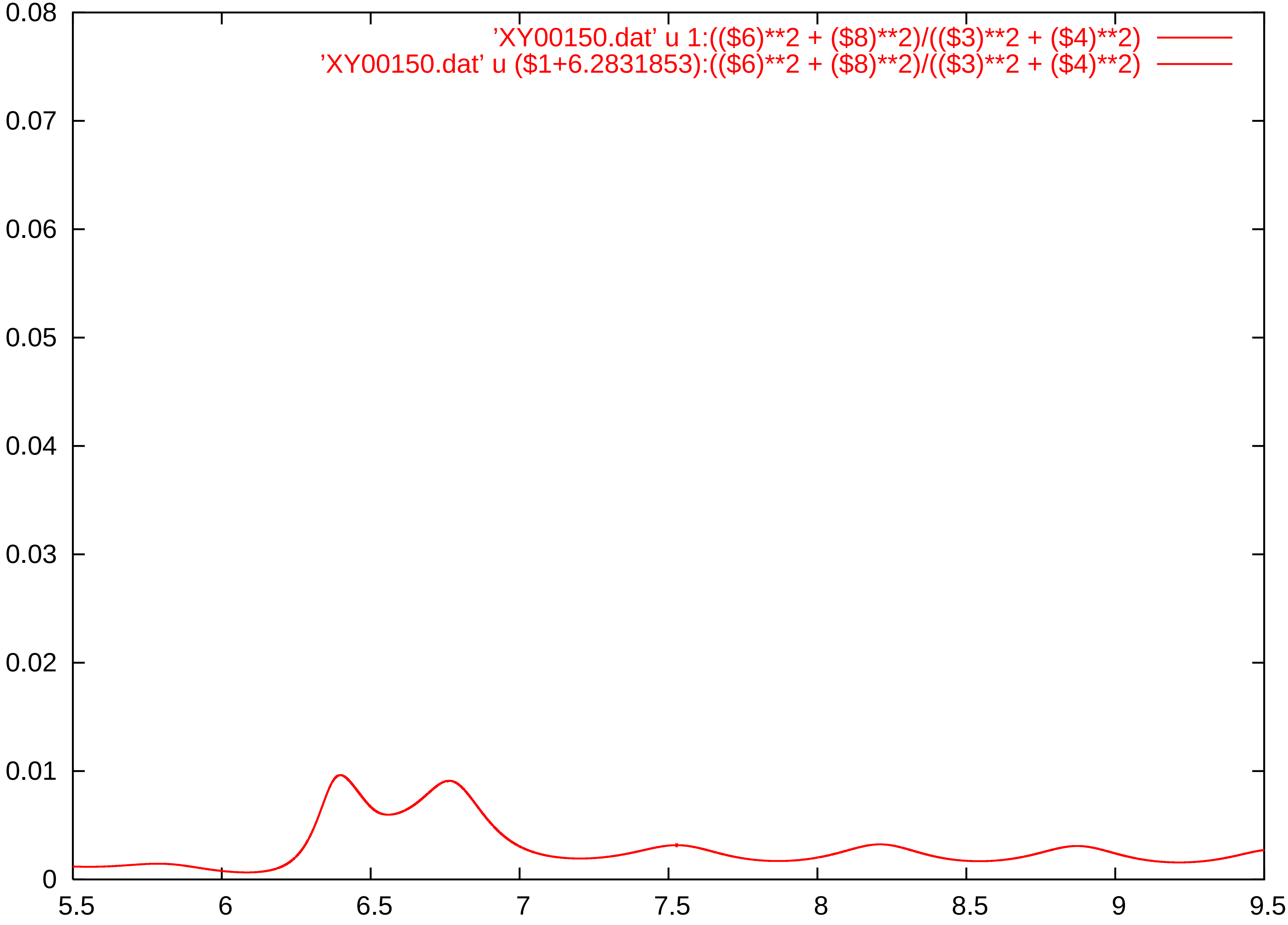

'XY00070.dat' u 1:(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
'XY00070.dat' u ($1+6.2831853):(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
5.5
6
6.5
7
7.5
8
8.5
9
9.5



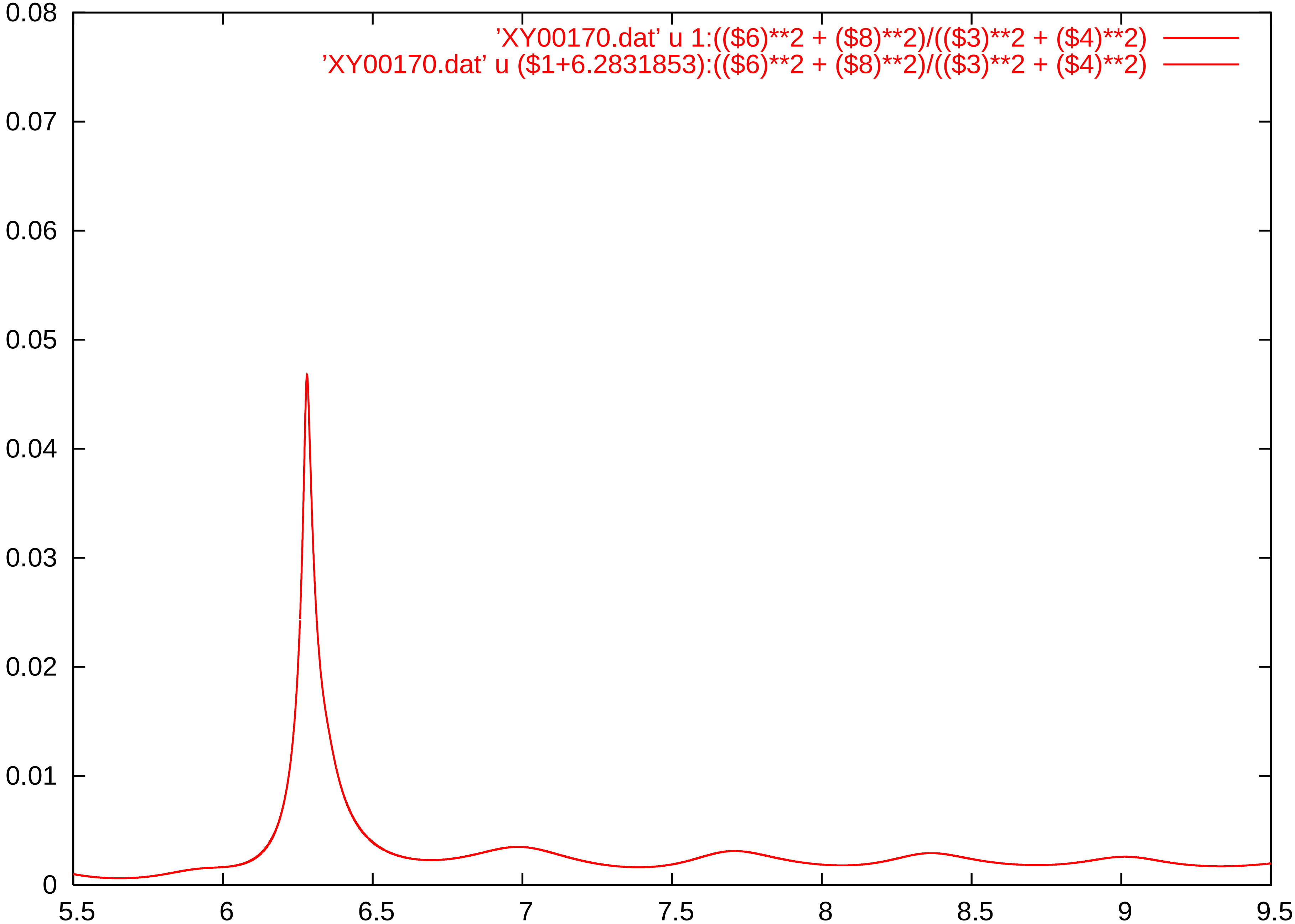
'XY00080.dat' u 1:(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
'XY00080.dat' u ($1+6.2831853):(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
5.5
6
6.5
7
7.5
8
8.5
9
9.5

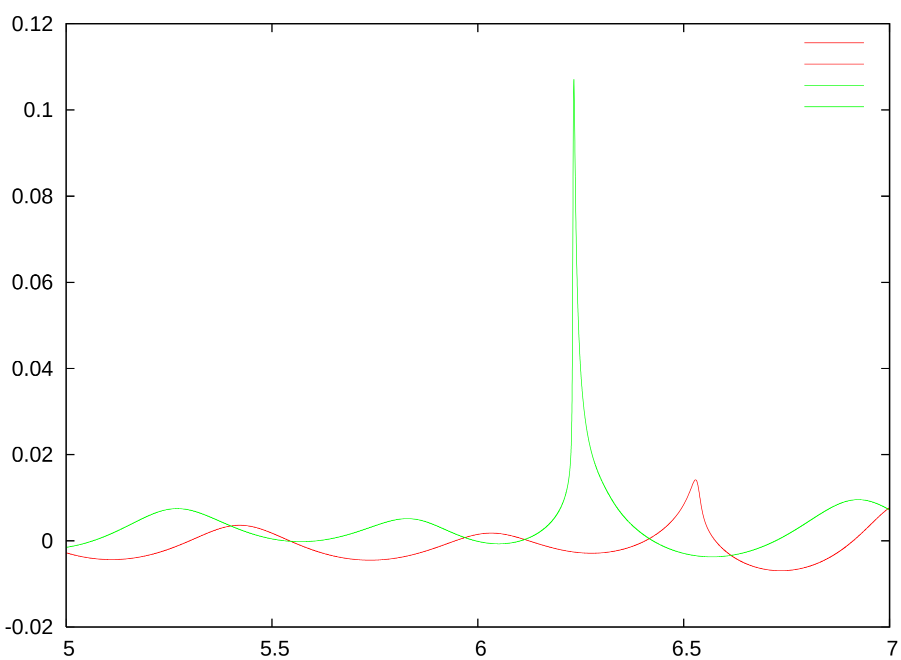


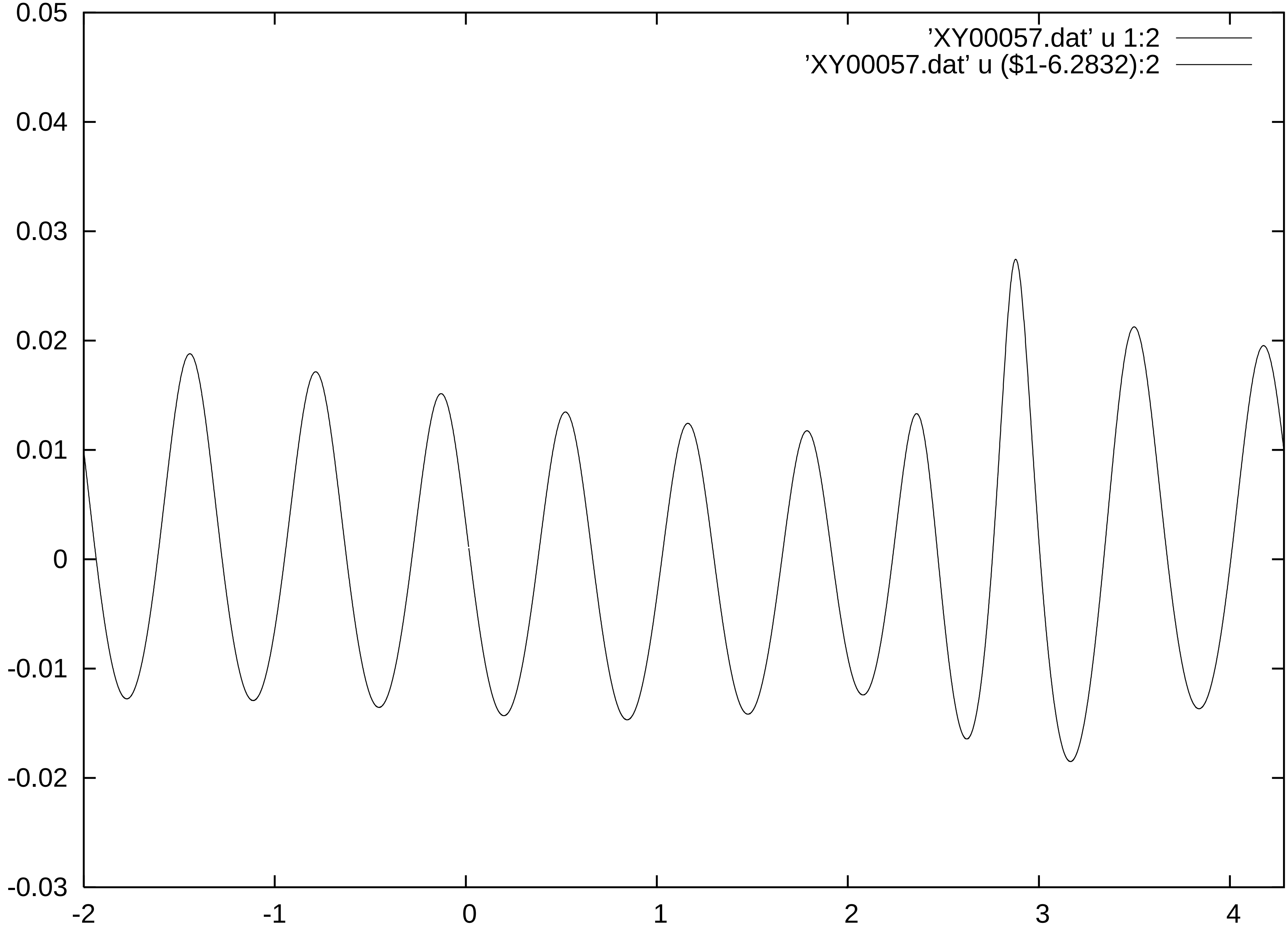
'XY00090.dat' u 1:(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
'XY00090.dat' u ($1+6.2831853):(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
5.5
6
6.5
7
7.5
8
8.5
9
9.5

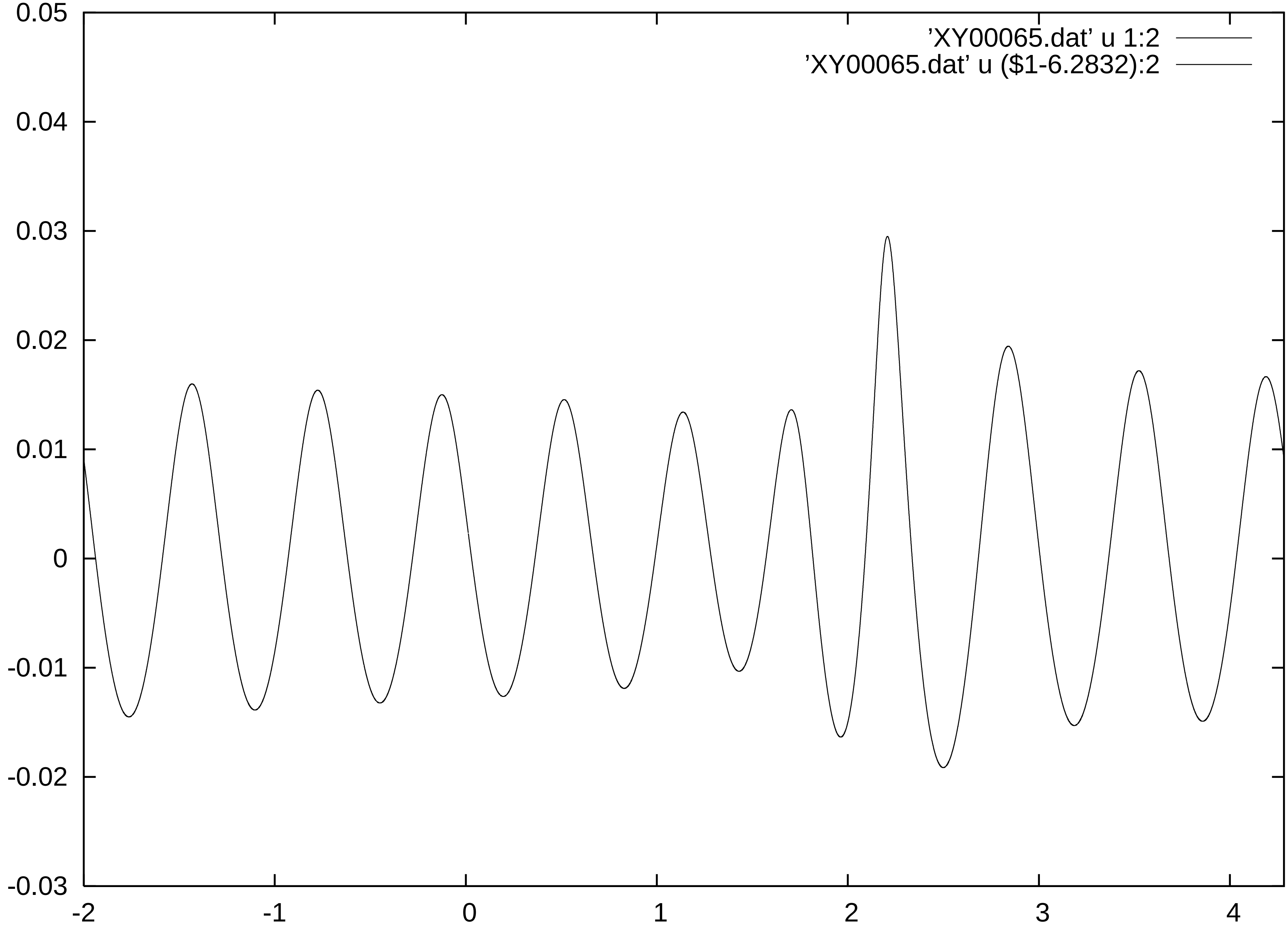


'XY00150.dat' u 1:(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
'XY00150.dat' u ($1+6.2831853):(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
5.5
6
6.5
7
7.5
8
8.5
9
9.5

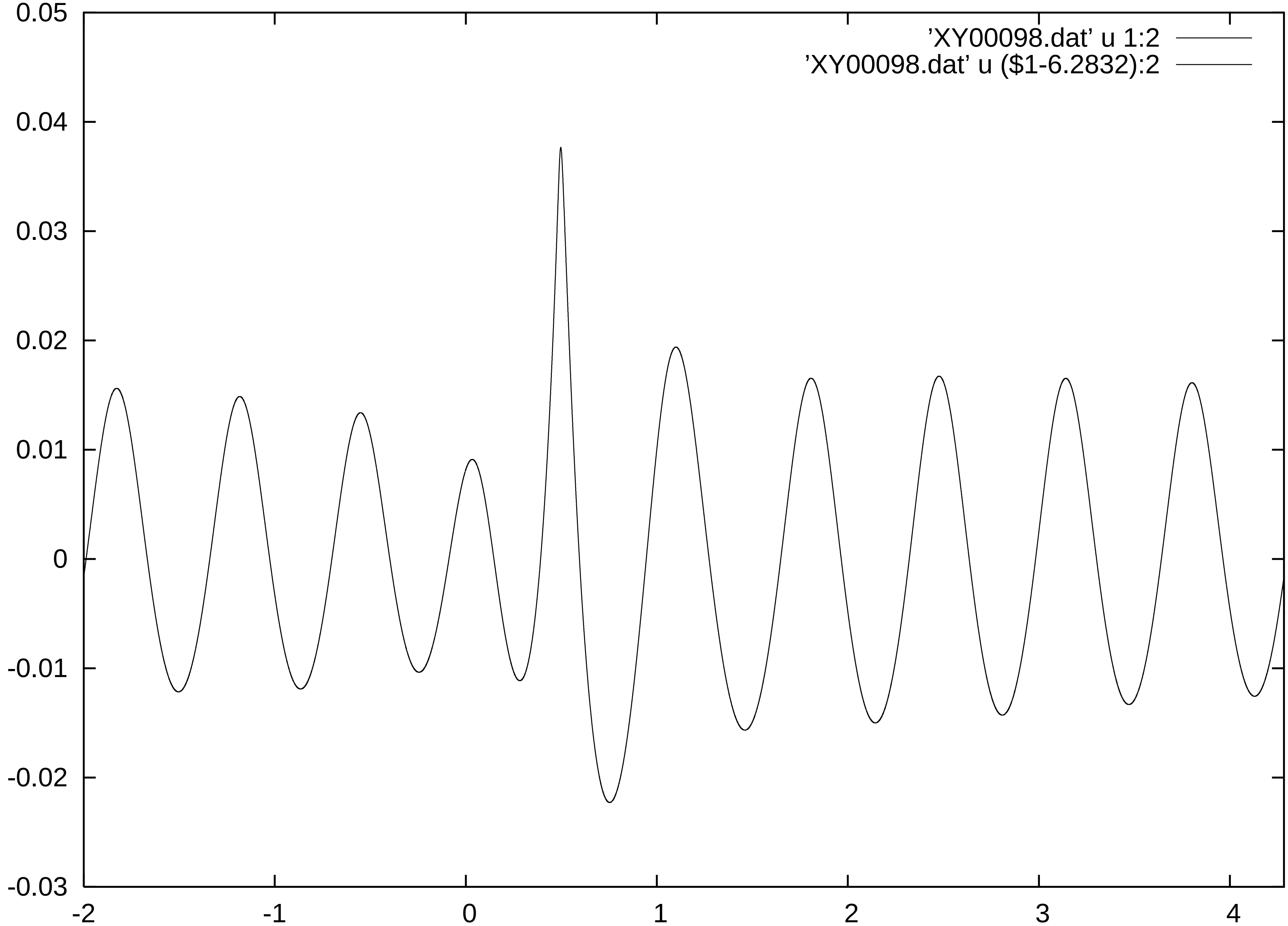

'XY00170.dat' u 1:(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
'XY00170.dat' u ($1+6.2831853):(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
5.5
6
6.5
7
7.5
8
8.5
9
9.5

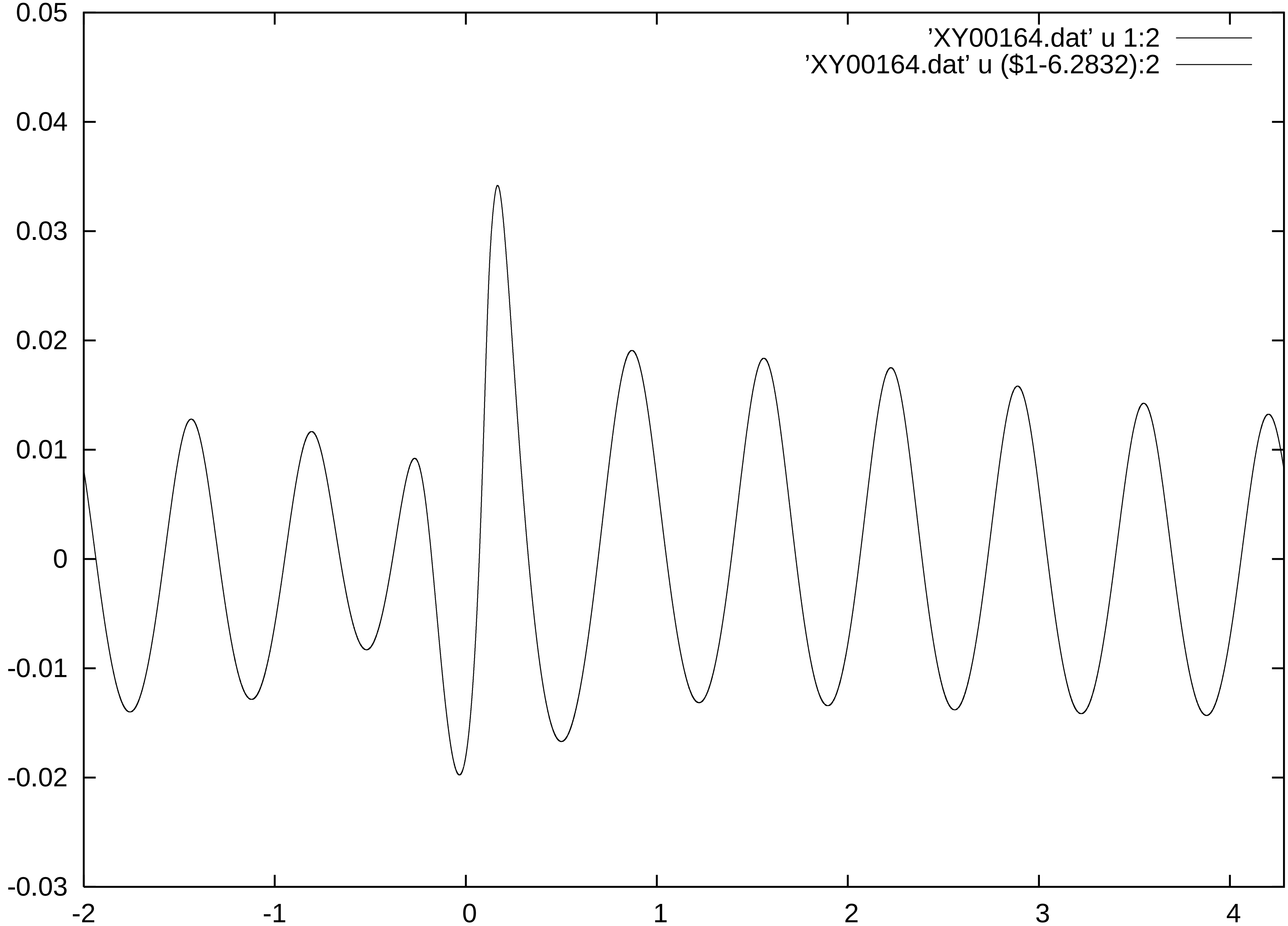

0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
5
5.5
6
6.5
7

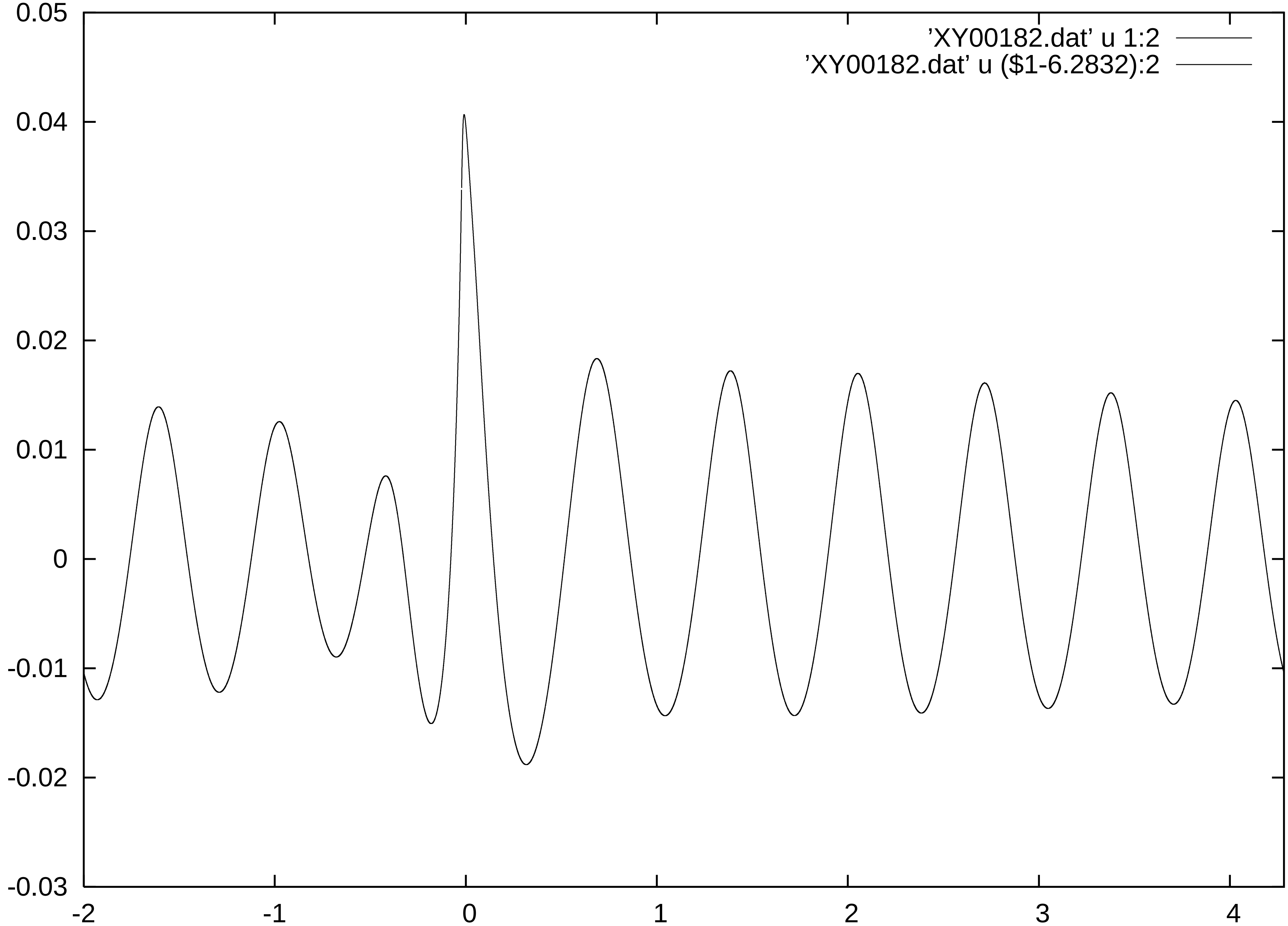

'XY00057.dat' u 1:2
'XY00057.dat' u ($1-6.2832):2
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-2
-1
0
1
2
3
4

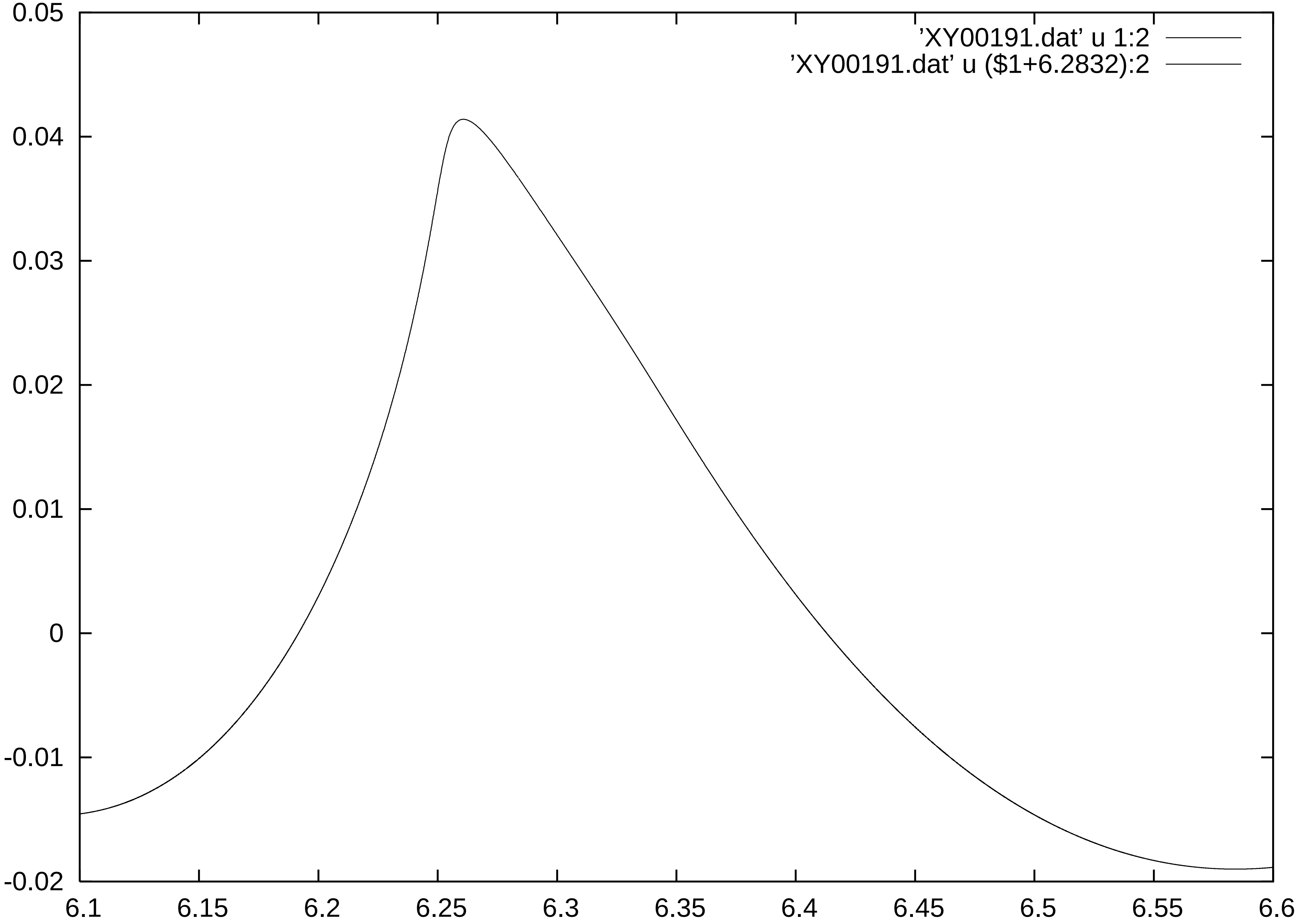


'XY00065.dat' u 1:2
'XY00065.dat' u ($1-6.2832):2
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-2
-1
0
1
2
3
4

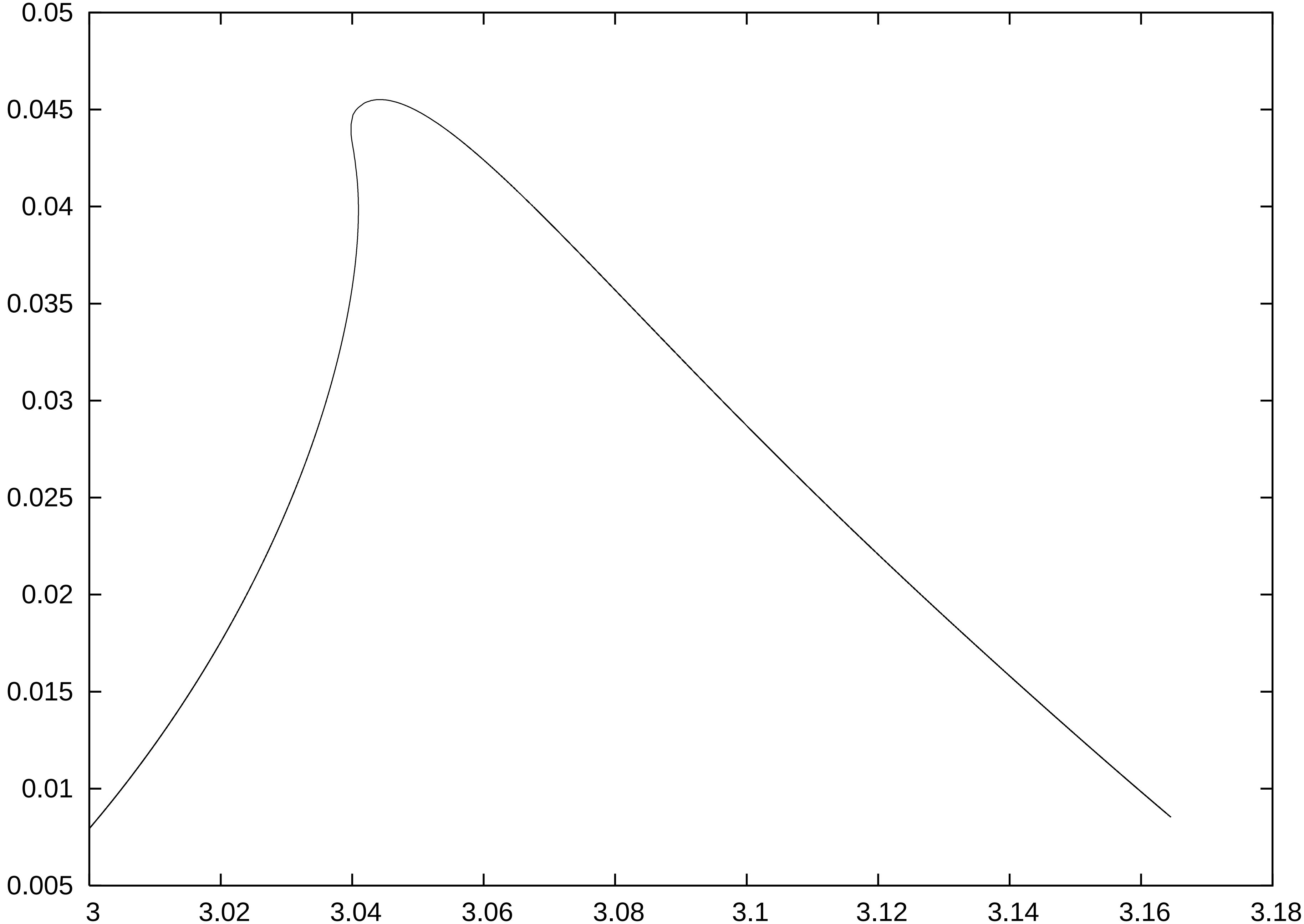

'XY00098.dat' u 1:2
'XY00098.dat' u ($1-6.2832):2
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-2
-1
0
1
2
3
4

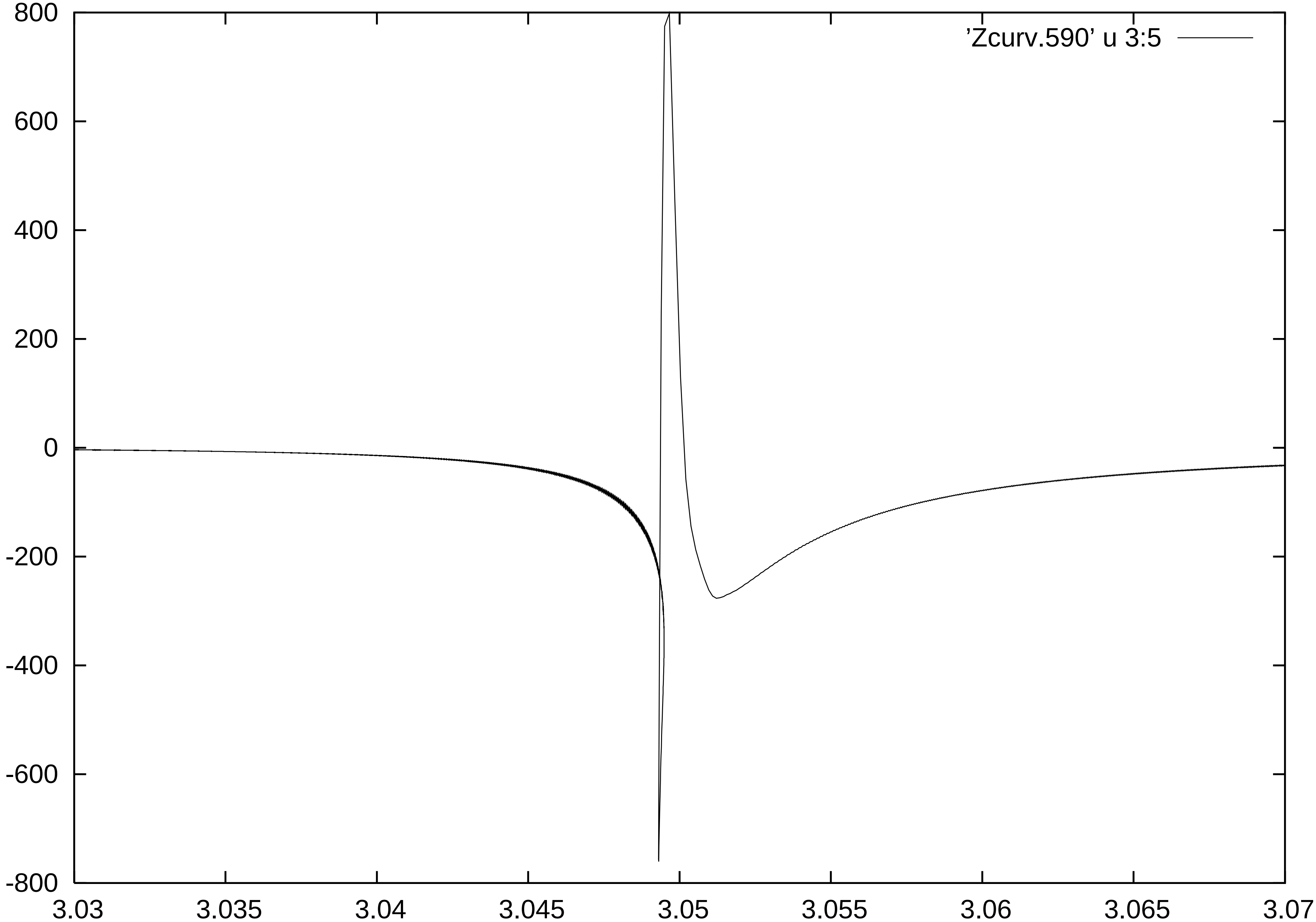

'XY00164.dat' u 1:2
'XY00164.dat' u ($1-6.2832):2
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-2
-1
0
1
2
3
4


’XY00182.dat’ u 1:2
’XY00182.dat’ u ($1-6.2832):2
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-2
-1
0
1
2
3
4

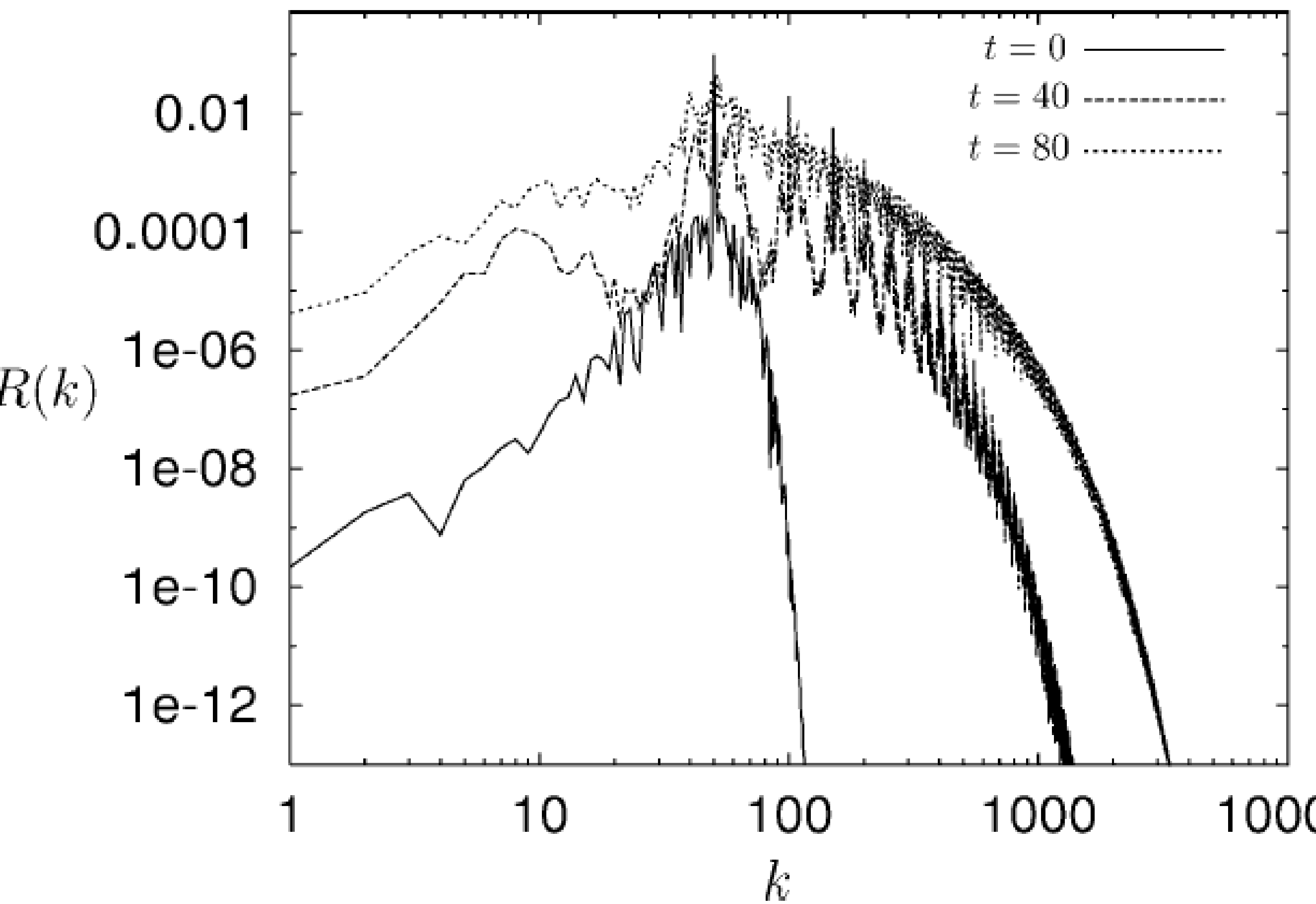


'XY00191.dat' u 1:2
'XY00191.dat' u ($1+6.2832):2
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
6.1
6.15
6.2
6.25
6.3
6.35
6.4
6.45
6.5
6.55
6.6

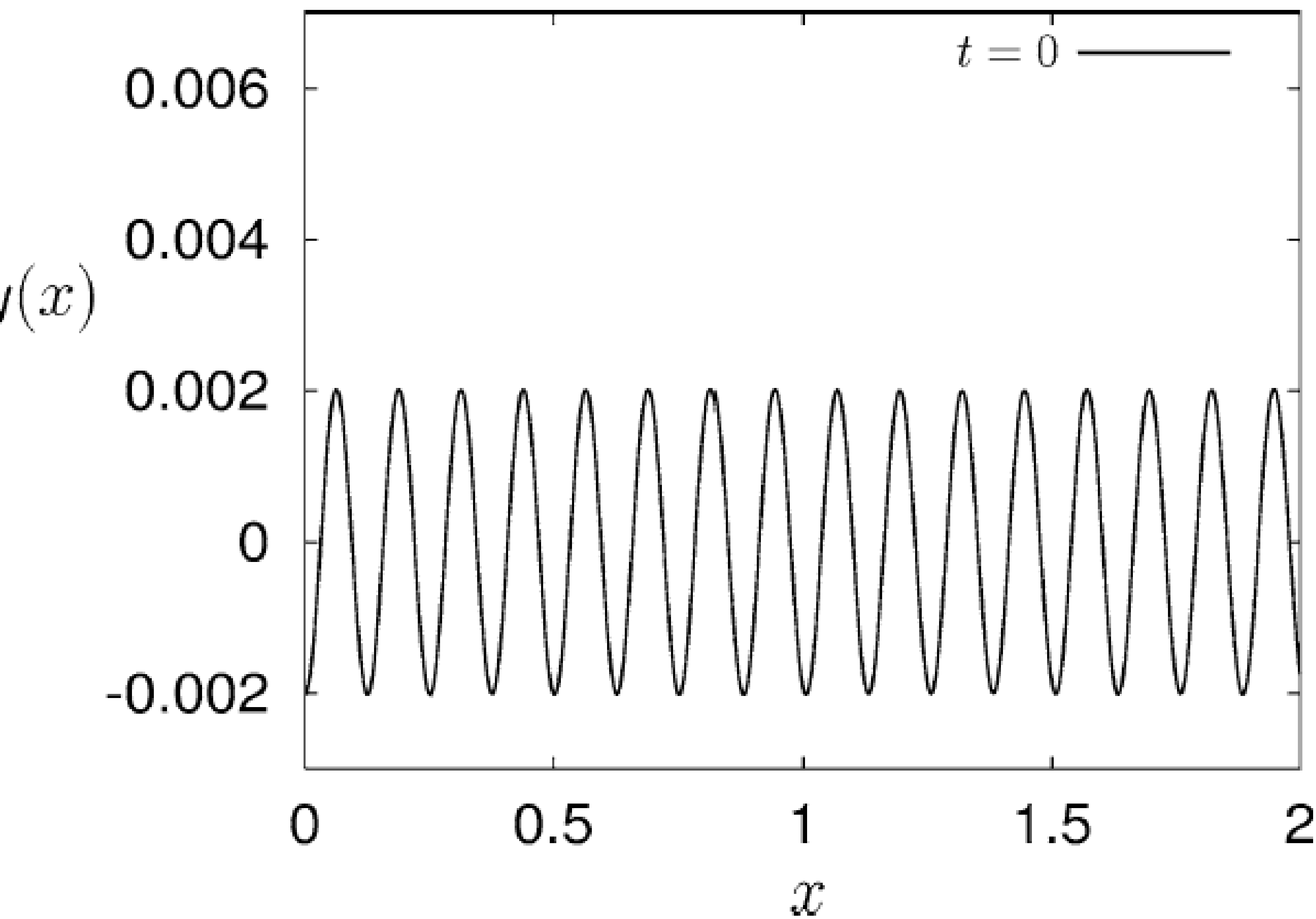

0.05
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
3
3.02
3.04
3.06
3.08
3.1
3.12
3.14
3.16
3.18



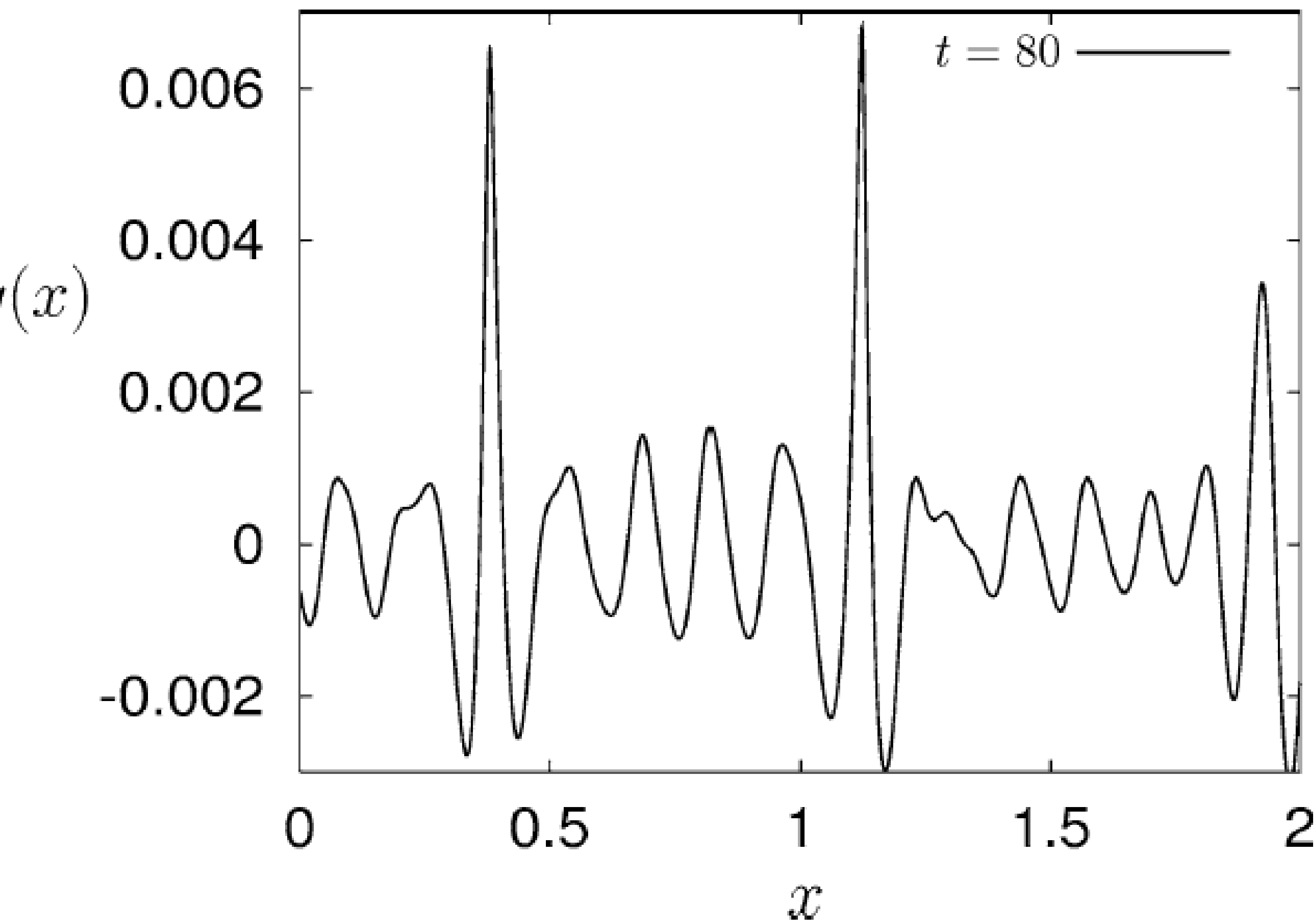
'Zcurv.590' u 3:5
800
600
400
200
0
-200
-400
-600
-800
3.03
3.035
3.04
3.045
3.05
3.055
3.06
3.065
3.07



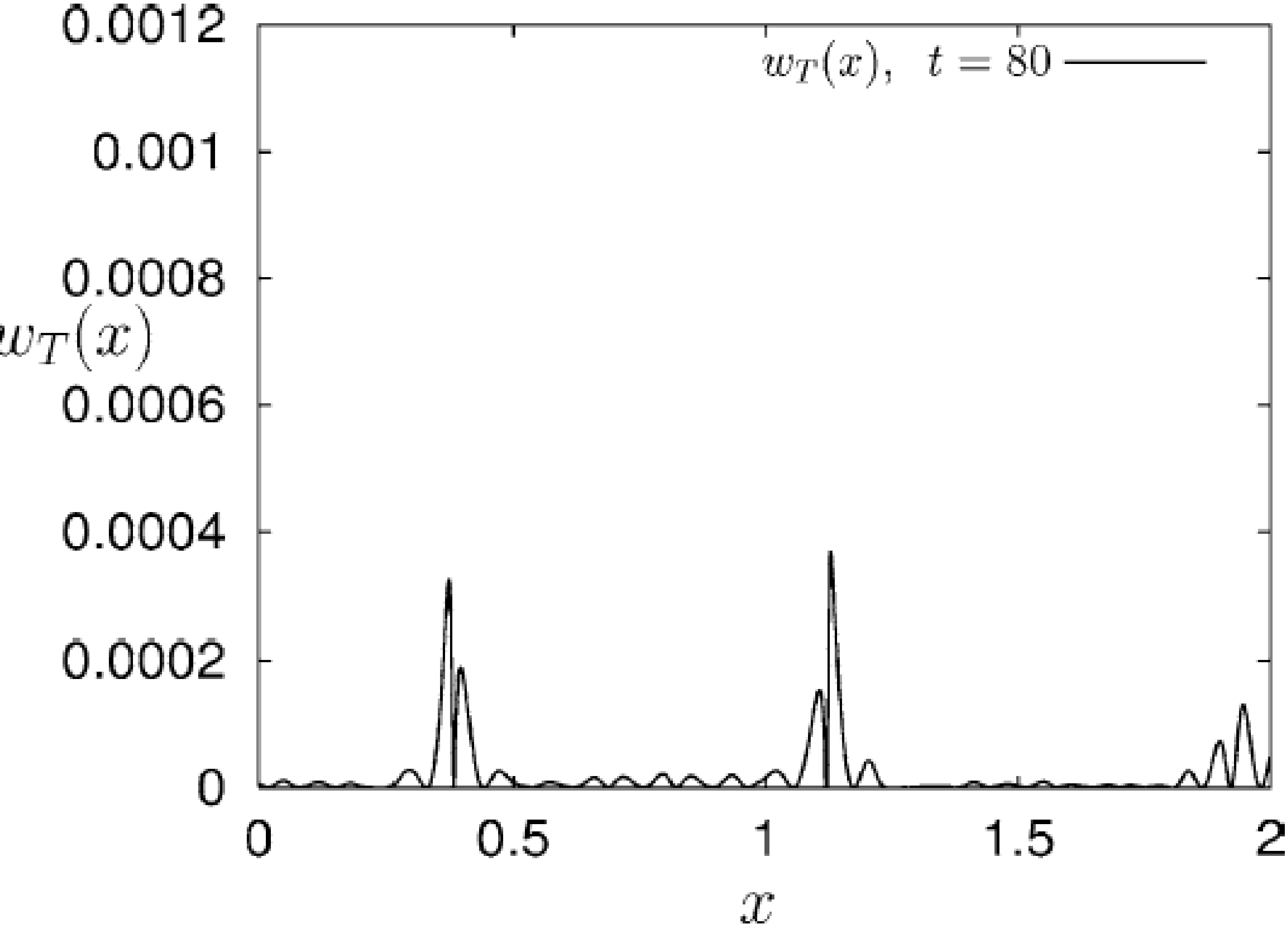
$t = 0$
$t = 40$
$t = 80$
0.01
0.0001
1e-06
1e-08
1e-10
1e-12
$R(k)$
1
10
100
1000
1000
k

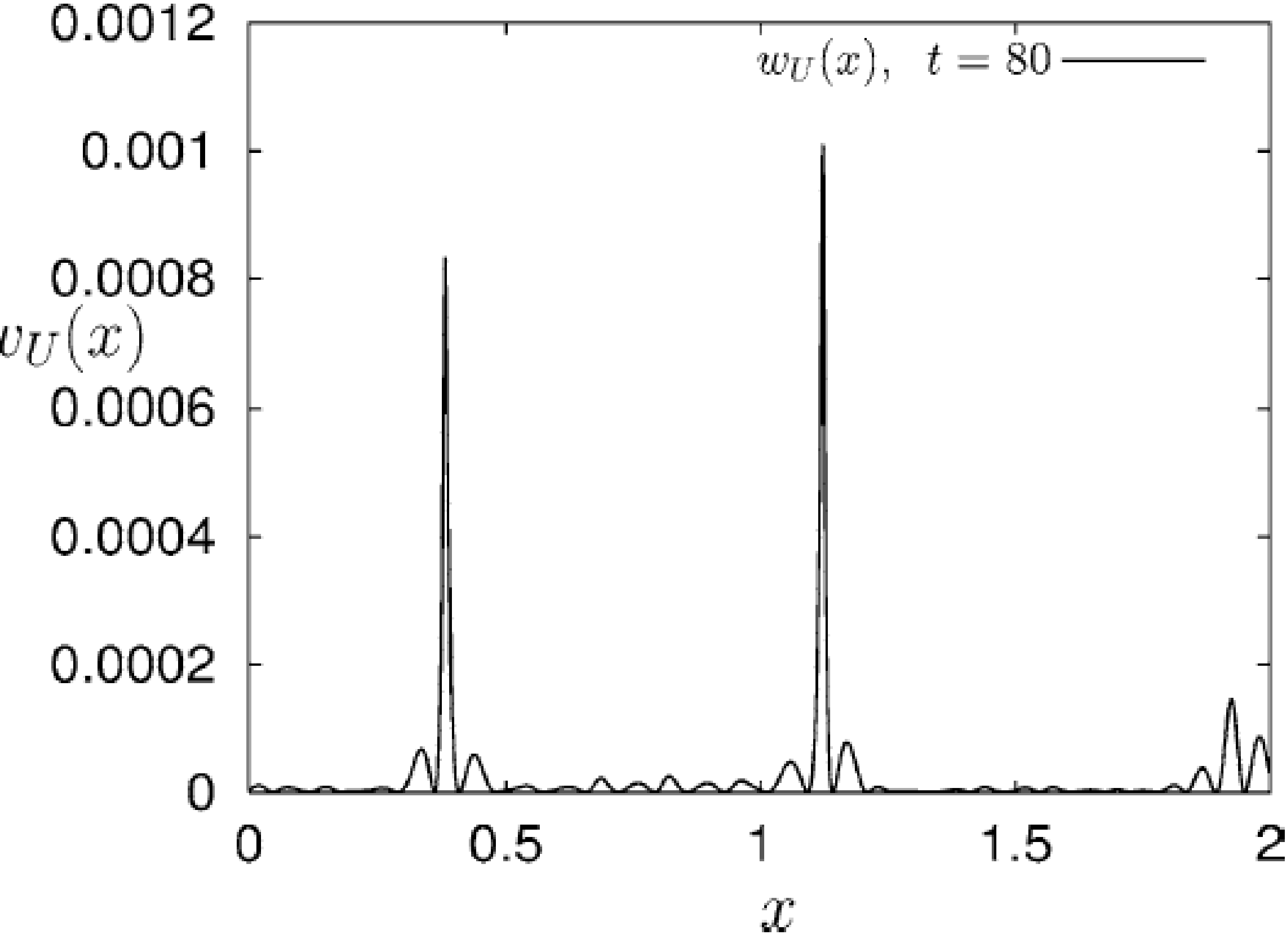


$t = 0$
0.006
0.004
$v(x)$
0.002
0
-0.002
0
0.5
1
1.5
2
x

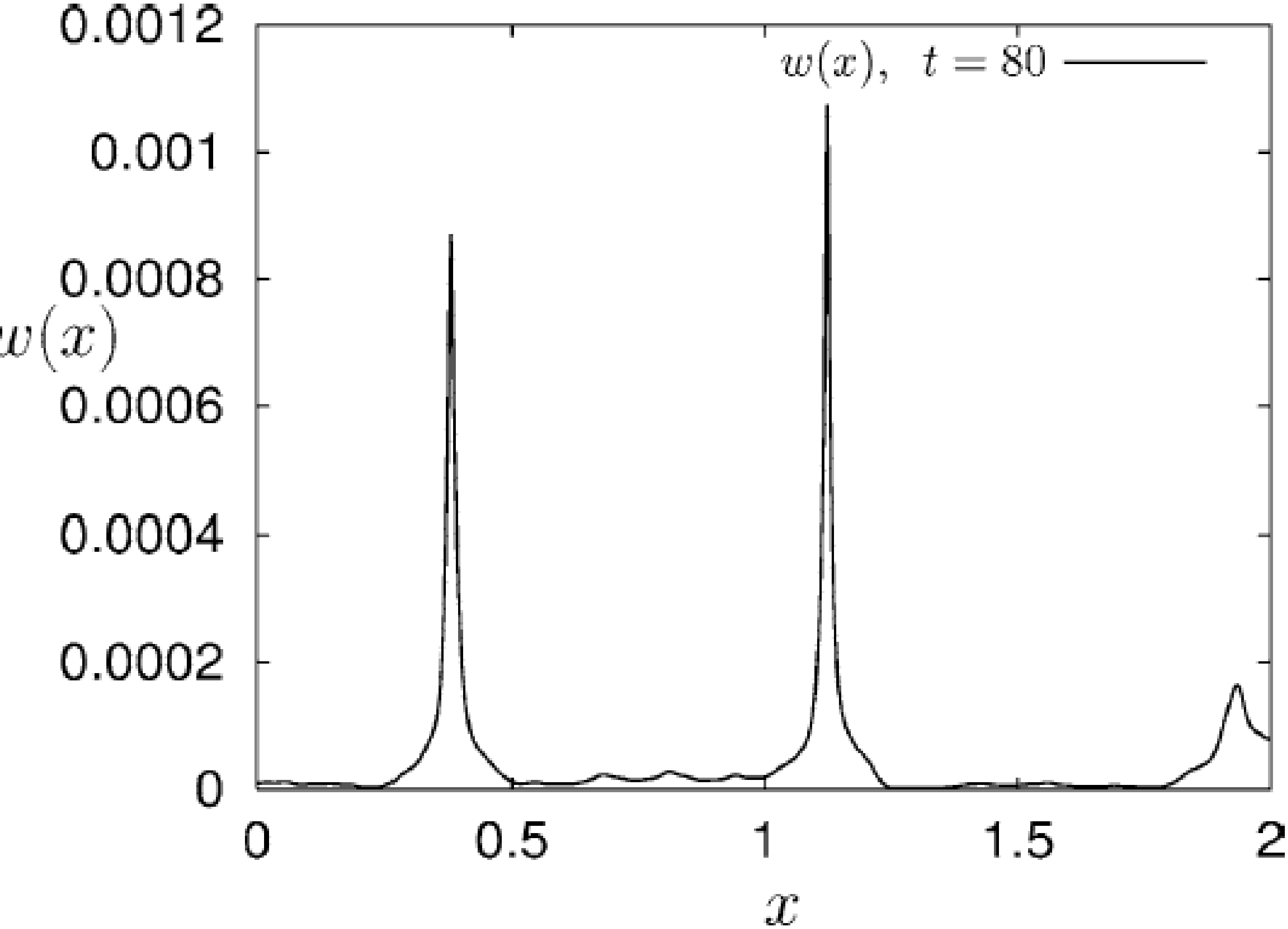


$t = 80$
0.006
0.004
0.002
0
-0.002
(x)
0
0.5
1
1.5
2
x

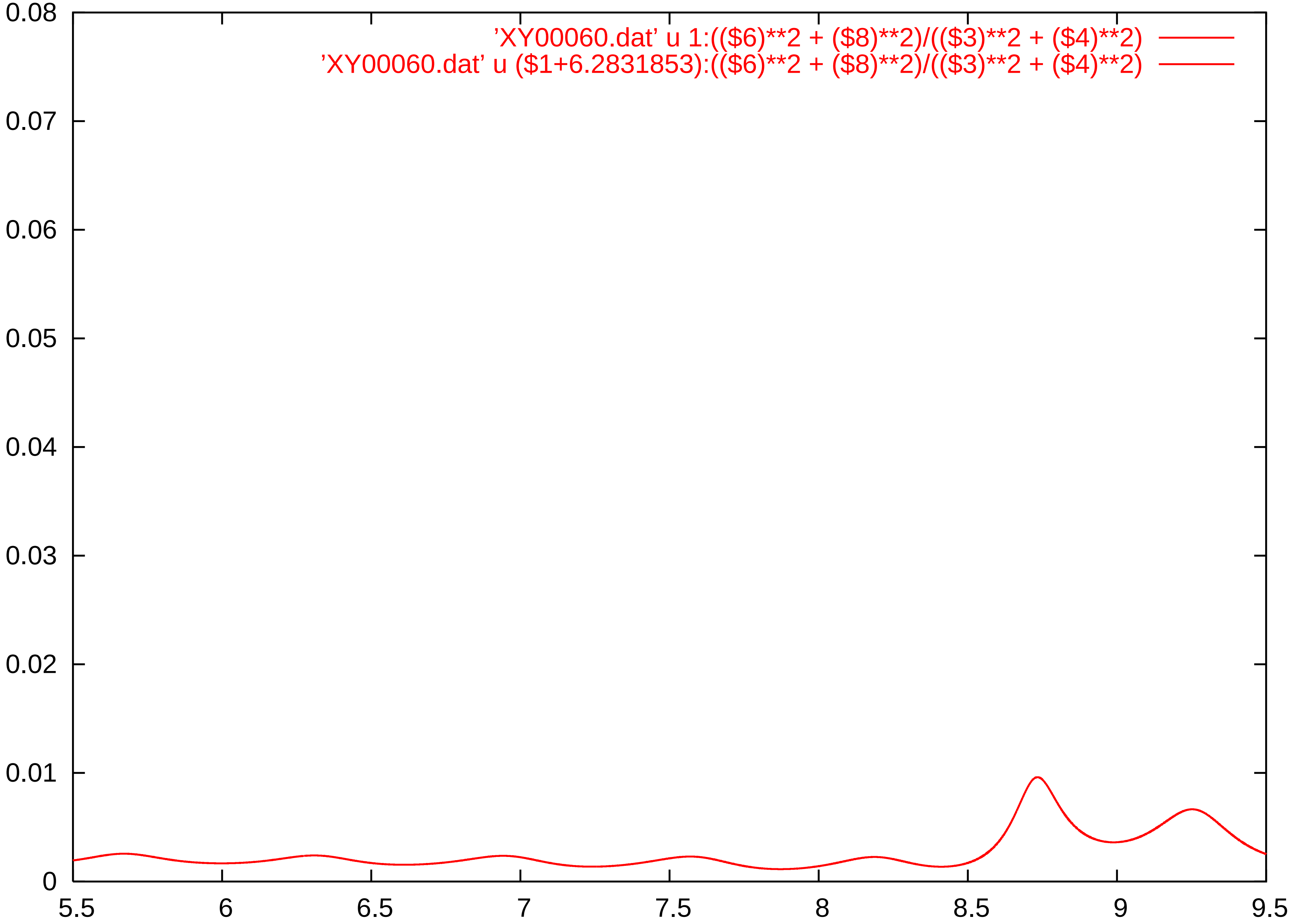

0.0012
0.001
0.0008
0.0006
0.0004
0.0002
0
$w_T(x)$
$w_T(x), \ \ t = 80$
0
0.5
1
1.5
2
x

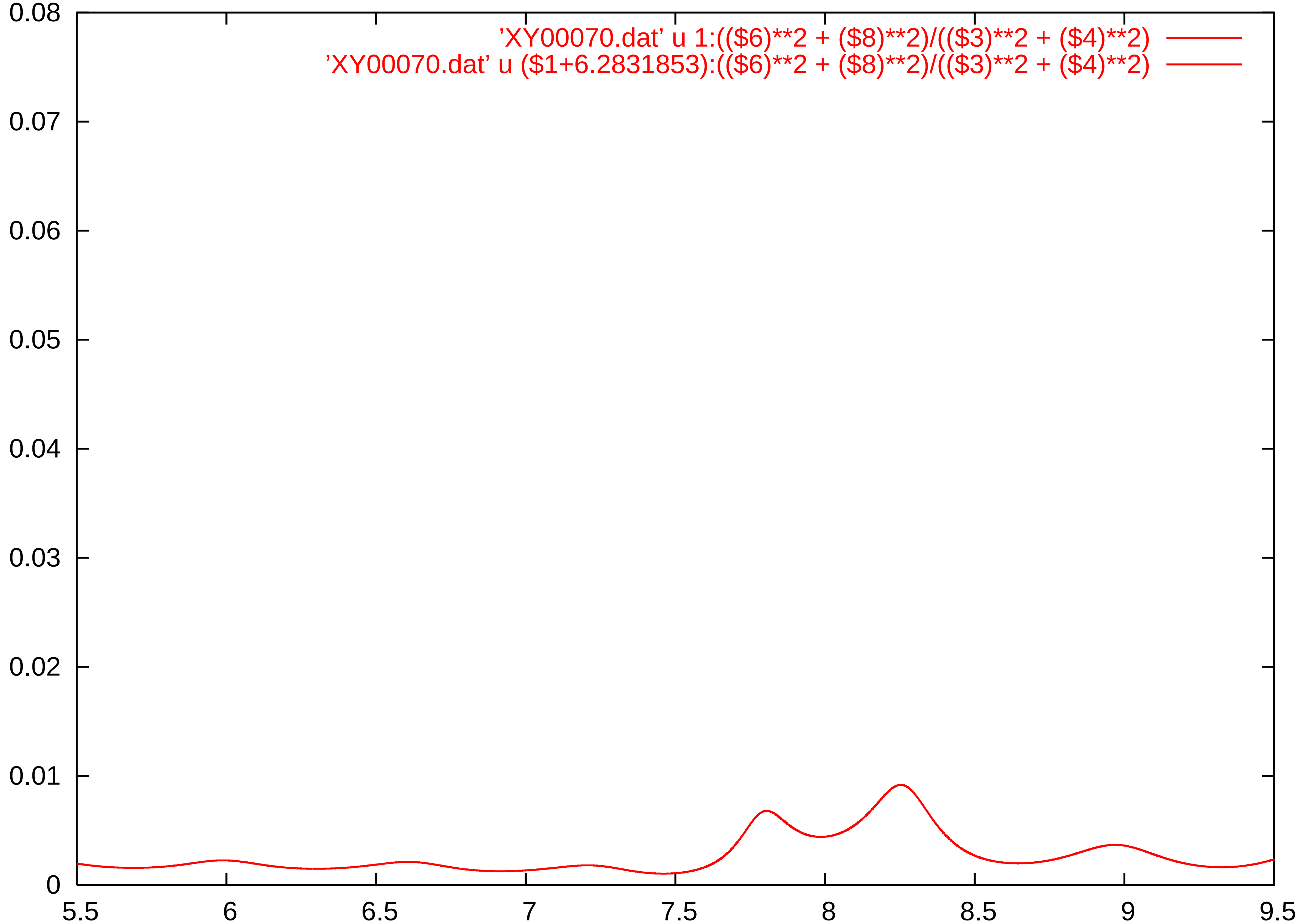


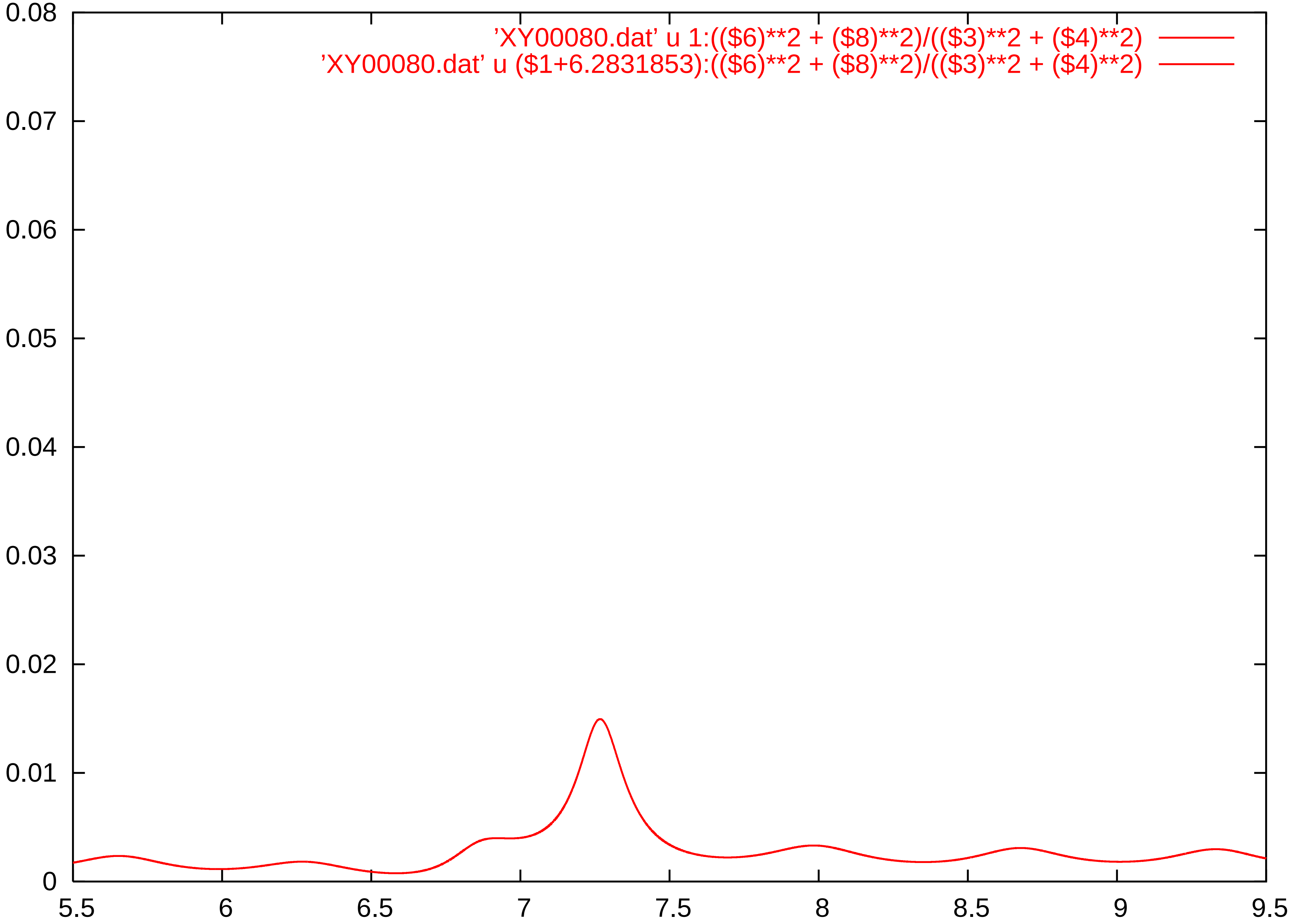
$w_U(x), \quad t = 80$
0.0012
0.001
0.0008
0.0006
0.0004
0.0002
0
$w_U(x)$
0
0.5
1
1.5
2
x



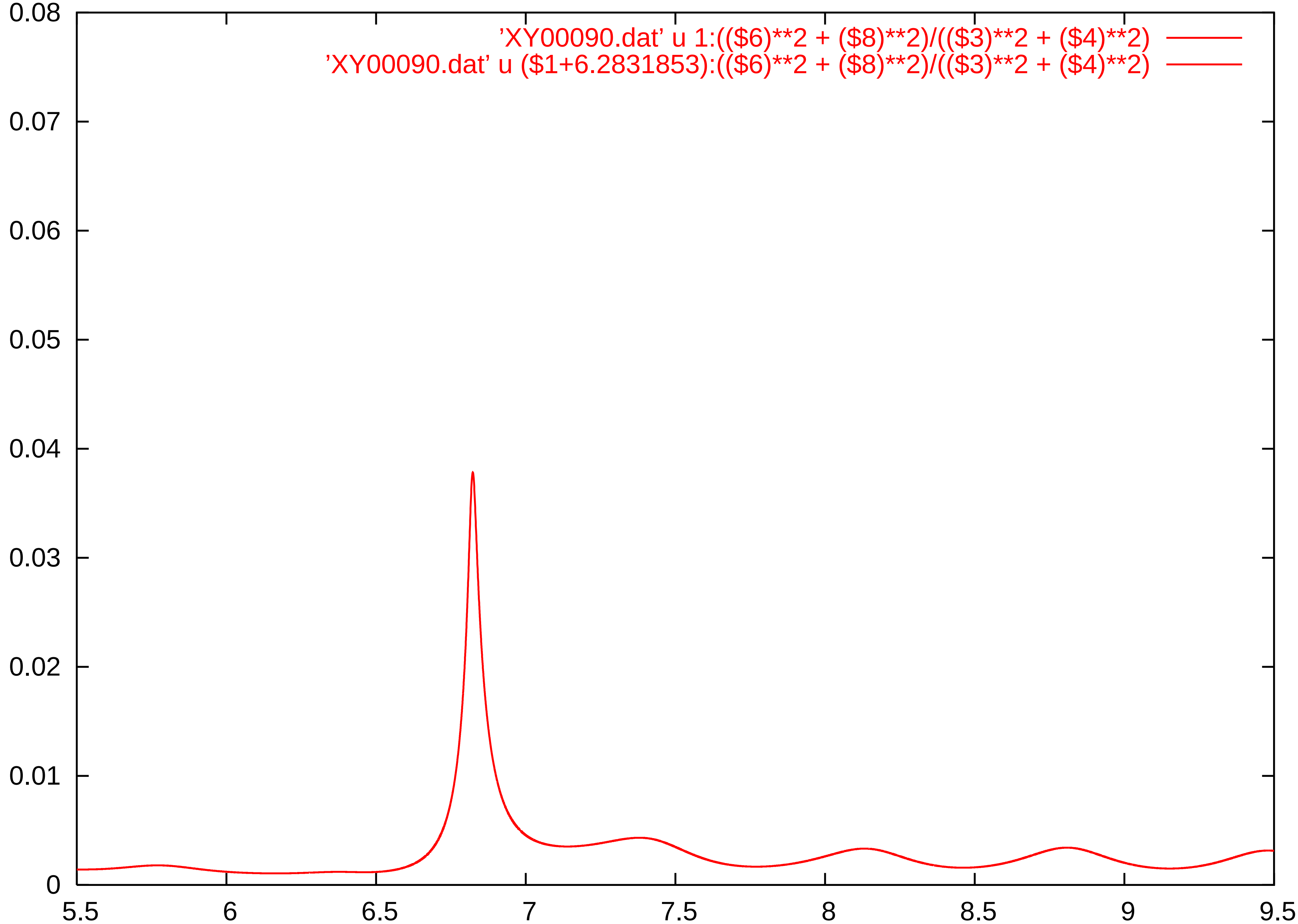
0.0012
0.001
0.0008
0.0006
0.0004
0.0002
0
$w(x)$
$w(x), \ \ t = 80$
0
0.5
1
1.5
2
x


'XY00060.dat' u 1:(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
'XY00060.dat' u ($1+6.2831853):(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
5.5
6
6.5
7
7.5
8
8.5
9
9.5

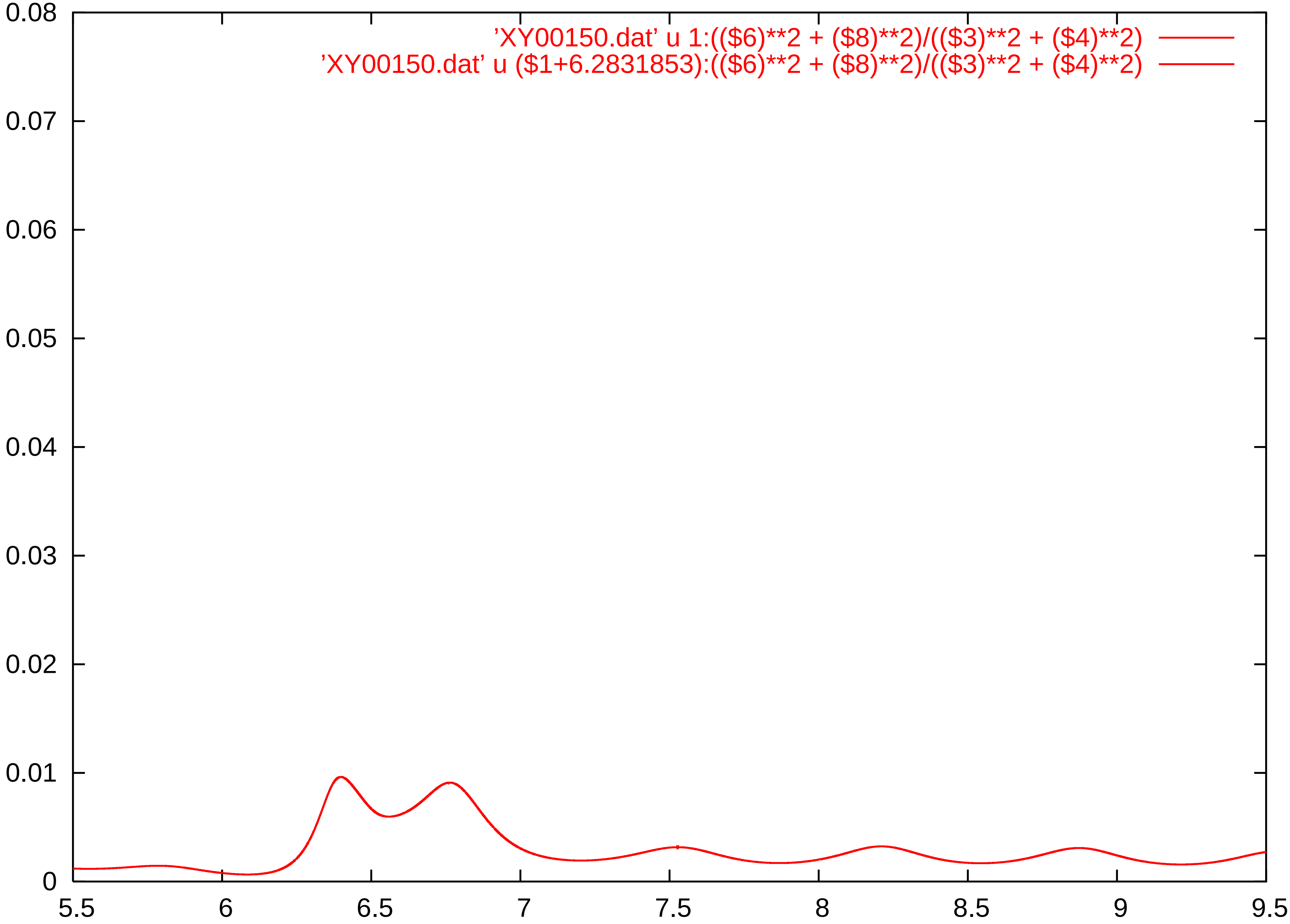

'XY00070.dat' u 1:(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
'XY00070.dat' u ($1+6.2831853):(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
5.5
6
6.5
7
7.5
8
8.5
9
9.5



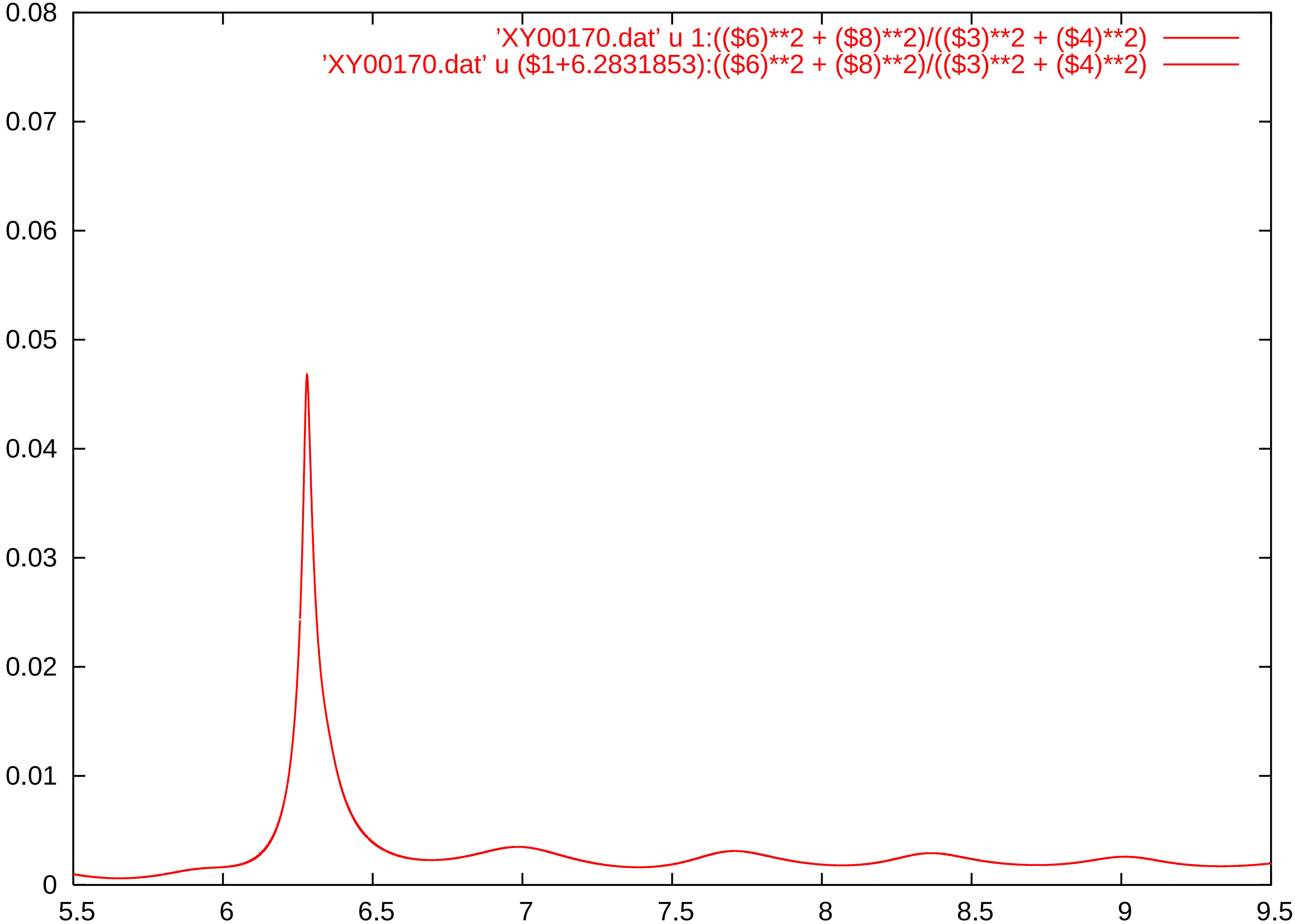
'XY00080.dat' u 1:(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
'XY00080.dat' u ($1+6.2831853):(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
5.5
6
6.5
7
7.5
8
8.5
9
9.5



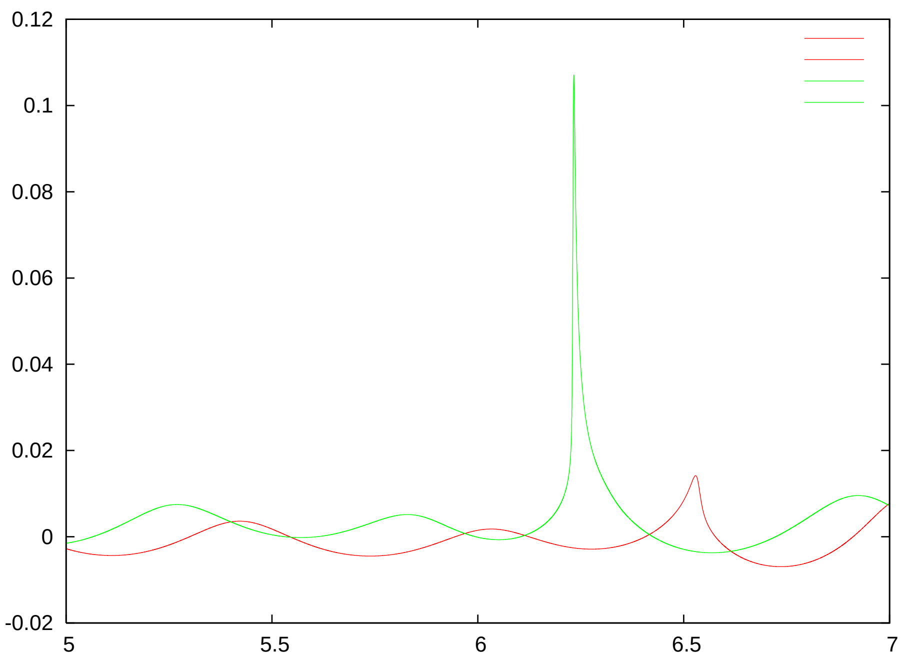
'XY00090.dat' u 1:(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
'XY00090.dat' u ($1+6.2831853):(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
5.5
6
6.5
7
7.5
8
8.5
9
9.5



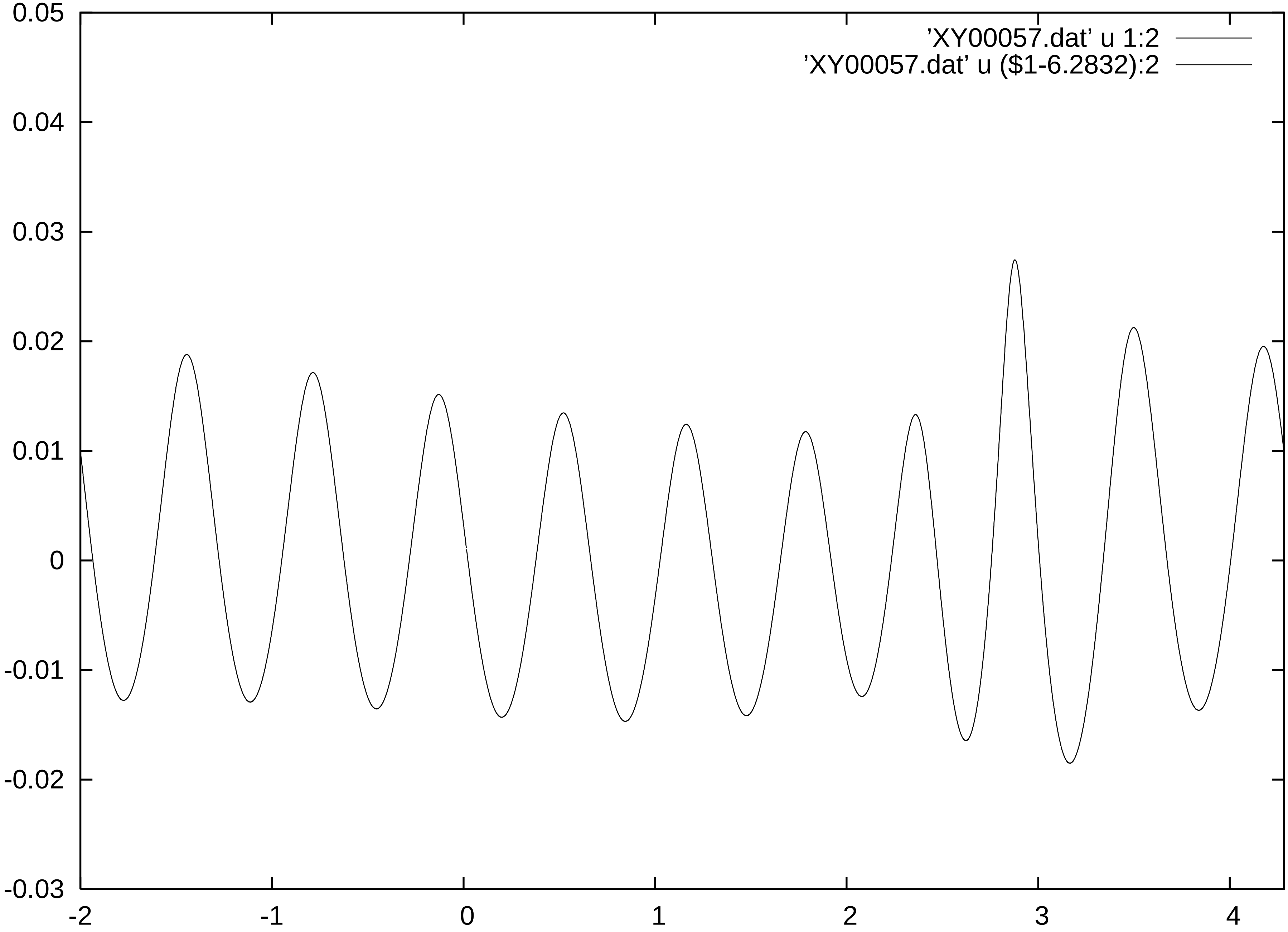
'XY00150.dat' u 1:(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
'XY00150.dat' u ($1+6.2831853):(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
5.5
6
6.5
7
7.5
8
8.5
9
9.5

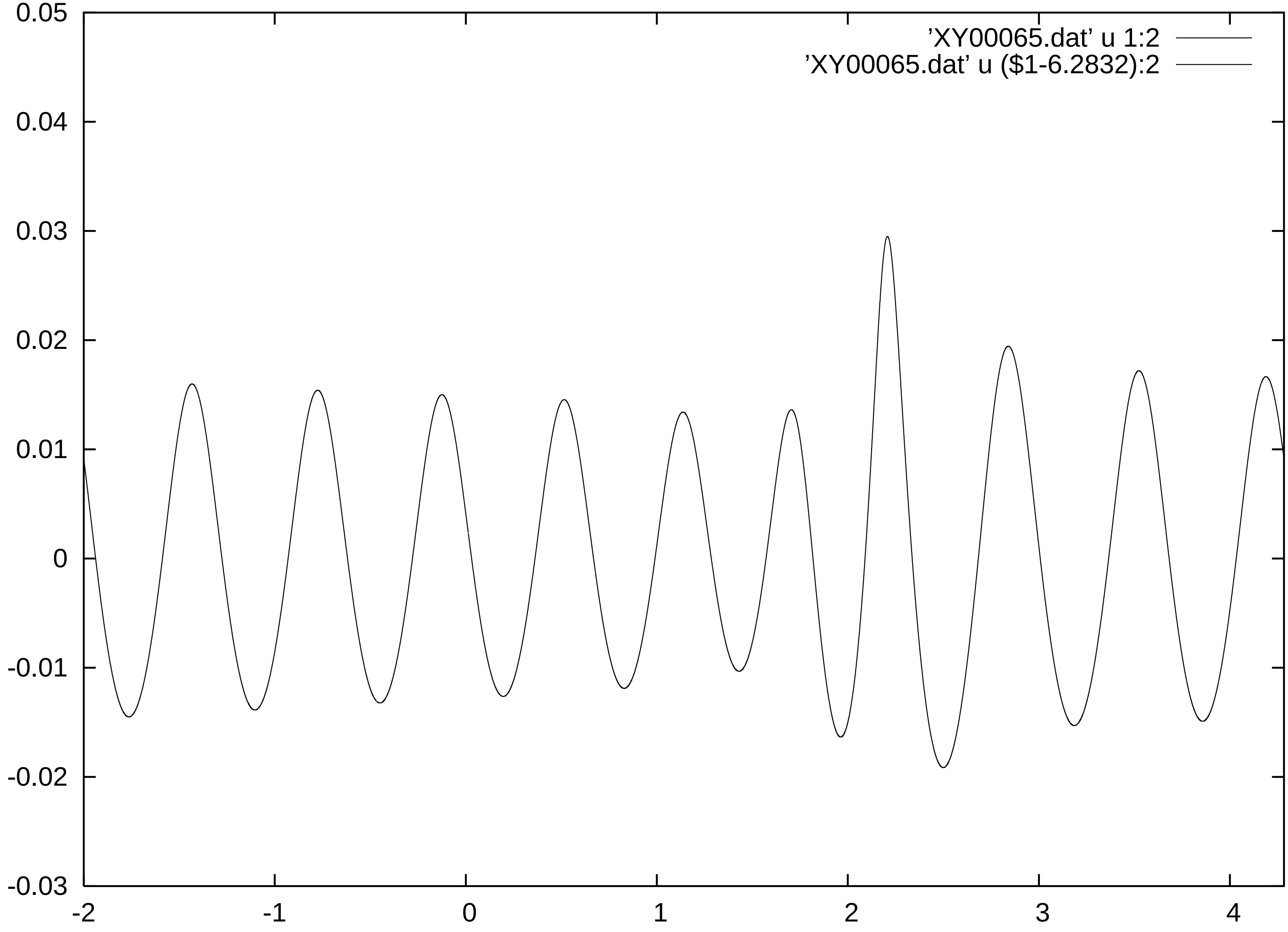


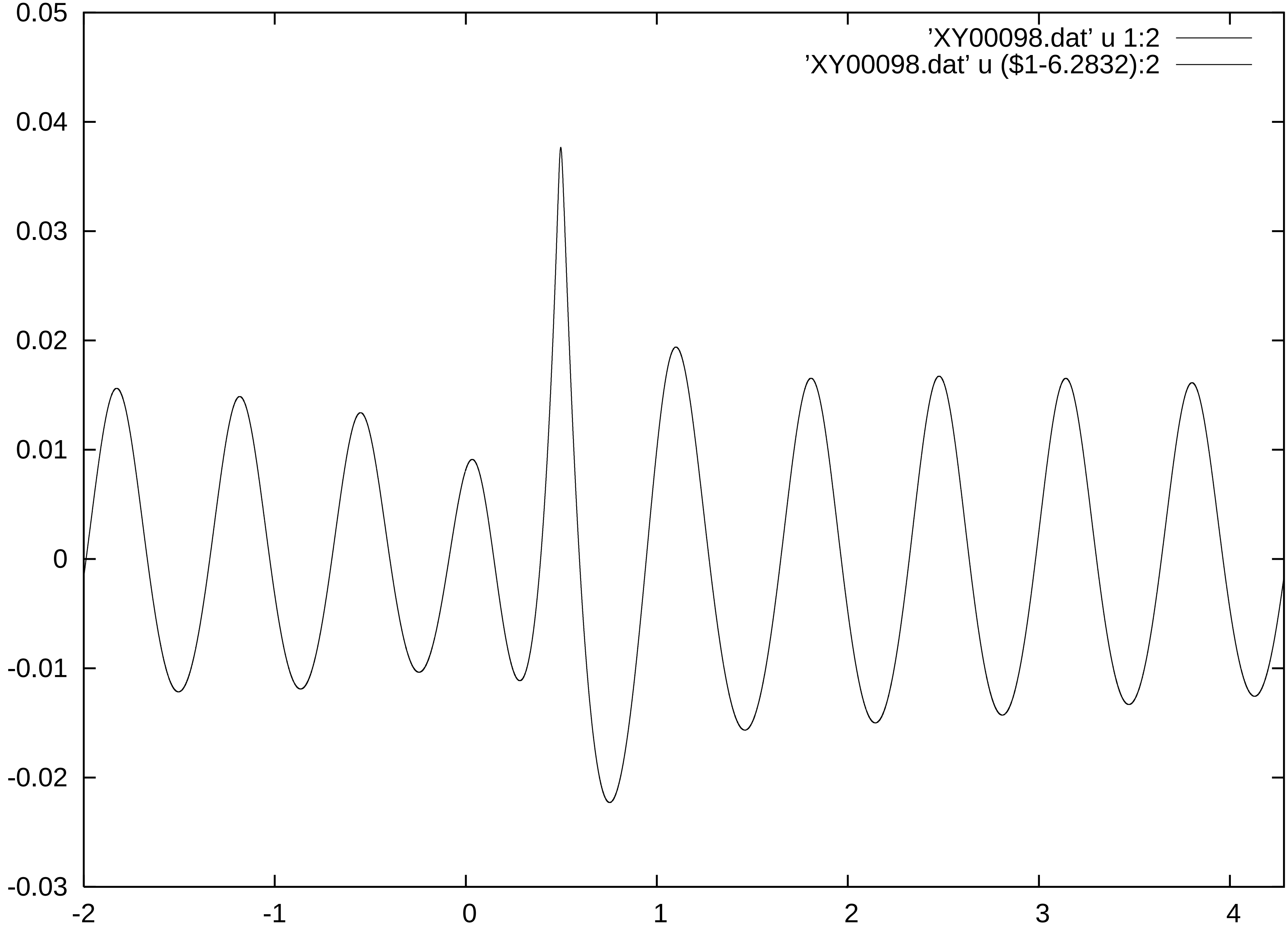
'XY00170.dat' u 1:(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
'XY00170.dat' u ($1+6.2831853):(($6)**2 + ($8)**2)/(($3)**2 + ($4)**2)
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
5.5
6
6.5
7
7.5
8
8.5
9
9.5

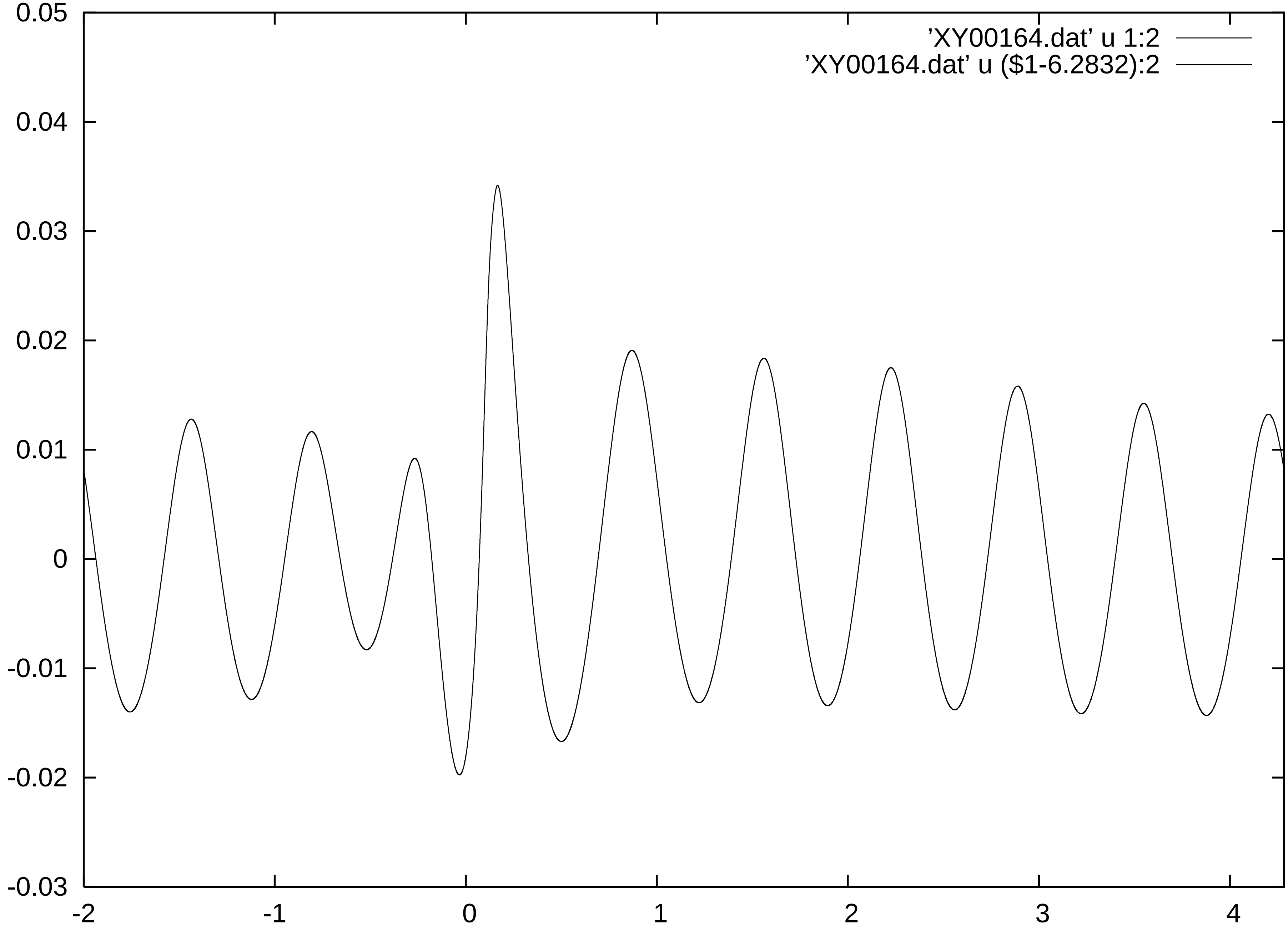


0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
5
5.5
6
6.5
7

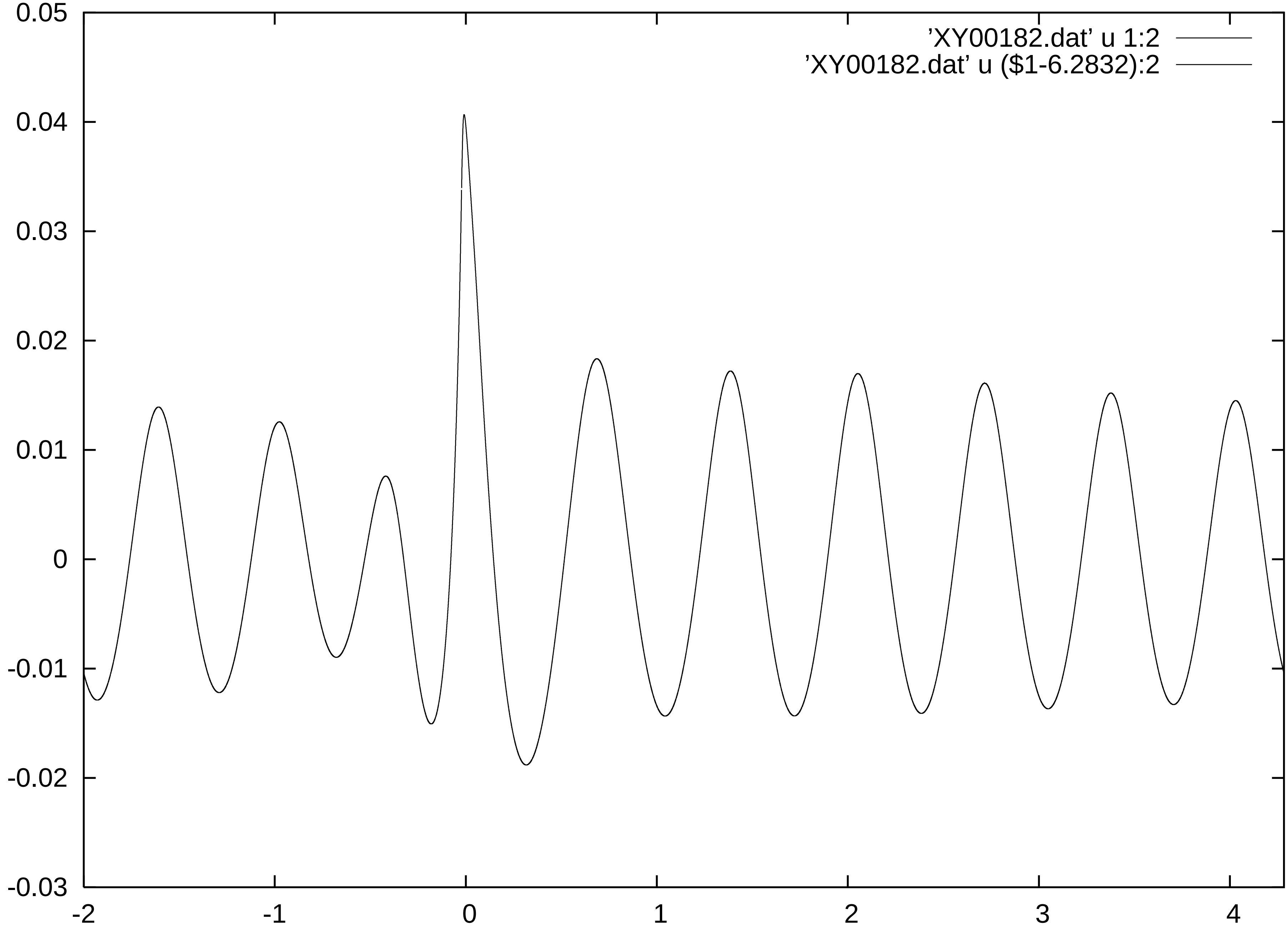

'XY00057.dat' u 1:2
'XY00057.dat' u ($1-6.2832):2
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-2
-1
0
1
2
3
4


'XY00065.dat' u 1:2
'XY00065.dat' u ($1-6.2832):2
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-2
-1
0
1
2
3
4

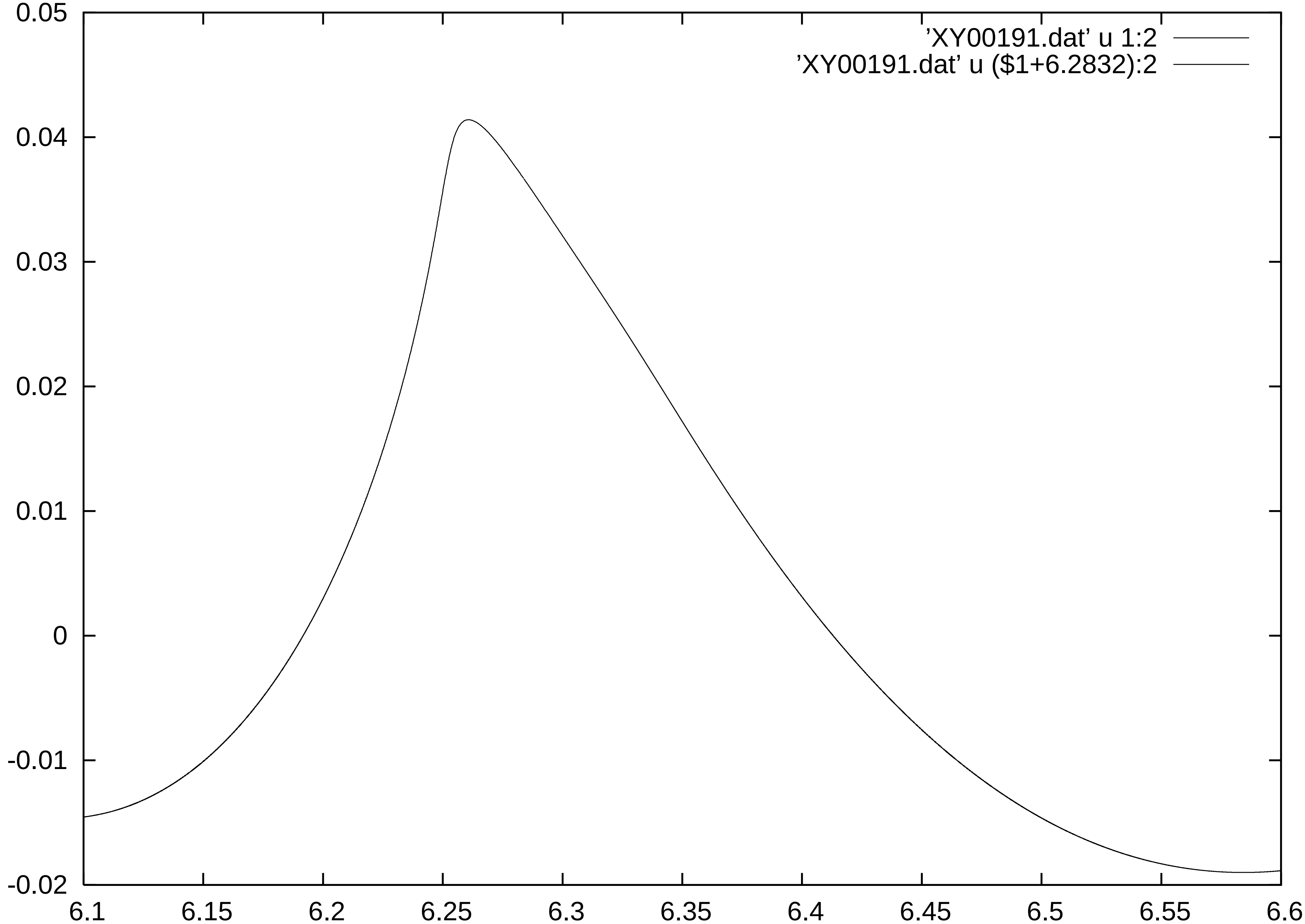

'XY00098.dat' u 1:2
'XY00098.dat' u ($1-6.2832):2
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-2
-1
0
1
2
3
4



'XY00164.dat' u 1:2
'XY00164.dat' u ($1-6.2832):2
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-2
-1
0
1
2
3
4



'XY00182.dat' u 1:2
'XY00182.dat' u ($1-6.2832):2
0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-2
-1
0
1
2
3
4



0.05
0.04
0.03
0.02
0.01
0
-0.01
-0.02
6.1
6.15
6.2
6.25
6.3
6.35
6.4
6.45
6.5
6.55
6.6
'XY00191.dat' u 1:2
'XY00191.dat' u ($1+6.2832):2