QUASI-RESONANT INTERACTIONS IN SHALLOW WATER

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I. INTRODUCTION

It is well known that a main ingredient in wave forecasting models is the nonlinear interactions that takes place among Fourier modes. The theory on these interactions, developed in the sixties independently by Zakharov [1] and Hasselmann [2], tells us that Fourier modes exchange energy only if four waves interact resonantly, i.e. if the following conditions must be satisfied:

$$\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4 = 0 \tag{1}$$

$$\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) - \omega(\mathbf{k}_3) - \omega(\mathbf{k}_4) = 0, \tag{2}$$

where $\mathbf{k}_i$ are wavenumbers and $\omega(\mathbf{k}_i)$ are angular frequencies related to the wavenumbers through the dispersion relation. This result is contained explicitly in the wave kinetic equation where two Dirac δ functions, one over wavenumbers and the other over frequency, appear in the multiple integrals of the nonlinear source term.

More recently it has been found [3] that in very particular situations of steep waves and narrow banded wave spectra, quasi-resonant interactions can also play an important role in the dynamics. This result has also been verified experimentally in wave tank facility [4]. By quasi-resonant interactions we mean that relation 2 is only approximately satisfied. From a mathematical point of view this requires a broadening of the δ dunction; more in particular in the wave kinetic equation derived in [3] the $\delta(\Delta\omega)$ is substituted with the following: $sin(\Delta\omega t)/\Delta\omega$ shown in figure 1. Note that for large time this function becomes very narrow and will tend to a Dirac δ function. Note that this result is obtained by a re-analysis of the derivation of the kinetic equation and has solid theoretical grounding.

These interesting results obtained in deep water have stimulated our interest and pushed us to apply similar concepts of quasi-resonant interactions to shallow water waves where it is well known that three-wave interactions may become important [5] even though they are never exactly resonating. Moreover, as will be discussed later, when waves run into very shallow water, or more precisely when the Ursell number becomes large, the standard kinetic equation in shallow water (see [6]) based on 4 wave resonant interactions is not applicable anymore

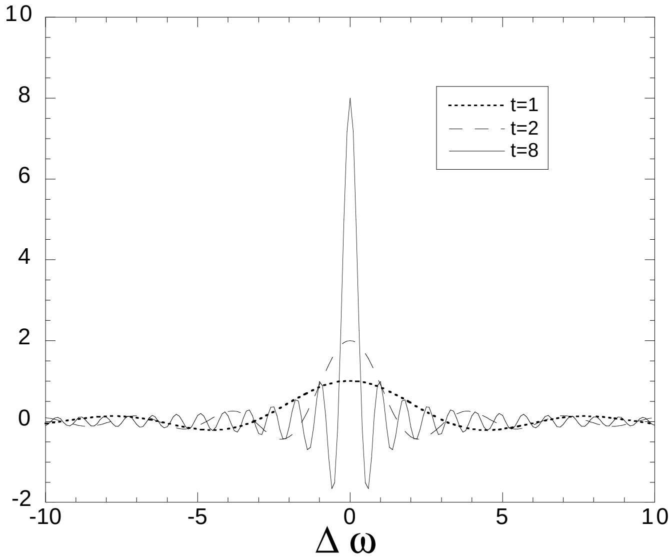

FIG. 1: Behaviour of the quasi-resonant function $Sin(\Delta\omega t)/\Delta\omega$ at different time. For large time the resonant function becomes very narrow and tends to a Dirac function.

and a different approach should be used in order to describe the stochastic evolution of surface gravity waves approaching the shore. Here our main goal is to derive an evolution equation for the wave action spectrum that takes into account three-wave interactions. The methodology is applied directly to the equations derived from the Hamiltonian formulation of water waves. Moreover in order to have some feeling on the results we will consider also the more intuitive case of the Korteweg-de Vries equation. Our choice is motivated by the fact that we intend in a future work to test with Montecarlo simulations the hypothesis of the quasi-gaussian approximation that is needed in order to derive the three-wave kinetic equation. In this framework, due to its numerically simple implementation, the Korteweg-de Vries equation offers a perfect starting point to understand the role of three-wave interactions from a statistical point of view. We mention that previous studies on the role of three-wave interactions have been made in the past, here we list a number of papers which also have stimulated the present work: [8–15].

The paper is organized as follows: in the next section we describe the most general deterministic equation valid for arbitrary depth which includes three and four-wave interactions. This equation, derived from Hamilton's principle, contains in the limit of shallow water, all the traditional long wave equations (Boussinesq, Korteweg de Vries, Miles). The limitation of the canonical transformation used to remove three-wave non resonant interactions is discussed. We will show that in the limit of shallow water the canonical transformation reduces exactly to the Stokes expansion in shallow water. The third Section contains the statistical description of the three-wave interactions equations.

II. HAMILTONIAN THREE AND FOUR-WAVE INTERACTION EQUATION

Starting from the Hamiltonian structure of Euler's equations for surface gravity waves and expanding the Hamiltonian for small steepness it is possible to show that the most general evolution equation that rules the dynamics of water waves in an inviscid and irrotational fluid is the following (see for example [7]):

$$\frac{\partial a_0}{\partial t} + i\omega_0 a_0 + i\int U^{(1)}_{0,1,2} a_1 a_2 \delta_{0-1-2} dk_{12} +$$
$$i2\int U^{(1)}_{2,1,0} a_1^* a_2 \delta_{01-2} dk_{12} + i\int U^{(3)}_{0,1,2} a_1^* a_2^* \delta_{012} dk_{12} +$$
$$i\int V^{(1)}_{0,1,2,3} a_1 a_2 a_3 \delta_{0-1-2-3} dk_{123} +$$
$$i\int V^{(2)}_{0,1,2,3} a_1^* a_2 a_3 \delta_{01-2-3} dk_{123} +$$
$$3i\int V^{(1)}_{3,2,1,0} a_1^* a_2^* a_3 \delta_{012-3} dk_{123} +$$
$$i\int V^{(4)}_{0,1,2,3} a_1^* a_2^* a_3^* \delta_{0123} dk_{123} + ... = 0. \quad (3)$$

Here the variable $a(k)$ is related to the surface elevation $\eta(k)$ and to the velocity potential calculated at the surface $\psi(k)$ in the following way:

$$\eta(k) = \sqrt{\frac{\omega(k)}{2g}}(a(k) + a^*(-k)) \quad (4)$$

$$\psi(k) = -i\sqrt{\frac{g}{2\omega(k)}}(a(k) - a^*(-k)), \quad (5)$$

where g is gravity acceleration. The coupling coefficients U and V are given in [7]. The compact notation in (3) is used for brevity; therefore for example the term δ_{012-3} means $\delta(k_0+k_1+k_2-k_3)$ and dk_{12} indicates $dk_1 dk_2$. The integrals are from $-\infty$ to $+\infty$. Equation (3) is written as a linear part plus a number of terms that include three-wave interactions and a number of terms that include four-wave interactions. Usually five and higher interactions are neglected for most of the applications and will not be considered here. Note that this equations contains both resonant and non resonant interactions.

The standard procedure to derive the so called Zakharov equation, which is the bases for the stochastic description of water waves, is to perform the canonical transformation, i.e. a quasi-identity transformation that maintains the Hamiltonian structure of the primitive equation. This transformation has the role of "removing" the three wave interactions. Note that those interactions are not really removed but only cast in such a way that they give a contribution to the four-wave interactions. From a mathematical point of view this transformation is possible only because three-wave interactions are never exactly resonant for water waves. The canonical transformation is expressed as a series expansion and for brevity we report it only up to second order:

$$a_0 = b_0 - \int \frac{U^{(1)}_{0,1,2} b_1 b_2 \delta_{0-1-2}}{\omega_0 - \omega_1 - \omega_2} dk_{12} -$$
$$2\int \frac{U^{(1)}_{2,1,0} b_1^* b_2 \delta_{01-2}}{\omega_0 + \omega_1 - \omega_2} dk_{12} - \int \frac{U^{(3)}_{0,1,2} b_1^* b_2^* \delta_{012}}{\omega_0 + \omega_1 + \omega_2} dk_{12} + ... \quad (6)$$

where $b(k)$ is the new canonical varible (the coefficients can be found in [7]). By applying the transformation (the third order term should be included (see [7])), to equation (3), we obtain the celebrated Zakharov equation which is the starting ingredient for building the kinetic equation for water waves:

$$\frac{\partial b_0}{\partial t} + i\omega_0 b_0 + i\int \tilde{V}^{(2)}_{0,1,2,3} b_1^* b_2 b_3 \delta_{01-2-3} dk_{123} = 0. \quad (7)$$

The canonical transformation appears as a quite complicated mathematical tool but it has a very deep physical meaning. In order to discuss it, here we find convenient to consider the shallow water limit; equation (4) becomes:

$$\eta(k) = \left(\frac{k^2 h}{g}\right)^{1/4} (a(k) + a^*(-k)) \quad (8)$$

and the kernels in the canonical transformation takes the following form:

$$\frac{U^{(1)}_{0,1,2}}{\omega_0 - \omega_1 - \omega_2} \sim \frac{U^{(3)}_{0,1,2}}{\omega_0 + \omega_1 + \omega_2} \sim \left(\frac{k^5}{g(kh)^{11}}\right)^{1/4} \quad (9)$$

Using these results it is possble to show that surface elevation in physical space is given by:

$$\eta(x) = \frac{1}{2}A(x)e^{ikx} + \frac{3}{8k^2h^3}A(x)^2 e^{2ikx} + ...c.c., \quad (10)$$

where the dependence on time has been neglected for brevity. Equation (10) is nothing but the Stokes series in shallow water which converges only if $Ur = ak/(kh)^3 << 1$, where Ur is the Ursell number. This conditions puts a severe limitation on the validity of the standard kinetic equation in shallow water. Therefore in section III, we propose a derivation of a kinetic equation starting directly from equation (3) without performing the canonical transformation (note that a similar approach is also discussed in [9]).

Relation between equation (3) and long wave equations

Before discussing the statistical theory, we show the generality of equation (3): we will derive from (3) well known long wave equations such as Boussinesq or Korteweg de Vries. We concentrate to the shallow water limit and therefore we will neglect the four-wave interactions and we will also assume one dimensional propagation in order to simplify the problem. In the limit of $kh \to 0$ the kernels of the equation simplify to:

$$U^{(1)}_{0,1,2} = U^{(1)}_{2,1,0} = U^{(3)}_{0,1,2} = \frac{3\sqrt{k_0 k_1 k_2}}{8\pi\sqrt{2}} \left(\frac{g}{h}\right)^{1/4} \qquad (11)$$

Using equations (4) and (5) and expanding the linear dispersion relation to second order in the limit of shallow water, equation (3) can be written as two equations, one for the surface elevation and the other for the velocity potential:

$$\frac{\partial \eta_0}{\partial t} - k^2 h\left(1 - \frac{1}{3}(kh)^2\right)\psi_0 + \qquad (12)$$
$$-\frac{1}{2\pi}\int k_0 k_1 \psi_1 \eta_2 \delta_{0-1-2} dk_{12} = 0,$$
$$\frac{\partial \psi_0}{\partial t} + g\eta_0 - \frac{1}{4\pi}\int k_1 k_2 \psi_1 \psi_2 \delta_{0-1-2} dk_{12} = 0.$$

If the inverse Fourier transform of the two equations is taken, one of the variants of the Boussinesq equations is obtained:

$$\frac{\partial \eta}{\partial t} + h\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{3}h^3 \frac{\partial^4 \psi}{\partial x^4} + \frac{\partial}{\partial x}\left(\eta \frac{\partial \psi}{\partial x}\right) = 0 \qquad (13)$$
$$\frac{\partial \psi}{\partial t} + g\eta + \frac{1}{2}\frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial x} = 0$$

As is well known, if waves are travelling in only one direction, the famous Korteweg- de Vries equation is obtained:

$$\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} + \frac{3c_0}{2h}\eta\frac{\partial \eta}{\partial x} + \frac{c_0 h^2}{6}\frac{\partial^3 \eta}{\partial x^3} = 0 \qquad (14)$$

with $c_0 = \sqrt{gh}$.

In the next section we will describe the statistical theory applied to equation (3); some attention will also be given to a direct application of the KdV equation.

III. STATISTICAL THEORY

Our aim here is to derive an evolution equation for the wave action N. In order to do so, we adopt the standard procedure, i.e. we first multiply equation (3) by a^*_j, then we write an evolution equation for the complex conjugate a^*_j, multiply it by a_i and add the two resulting equations. We then take an ensemble average of the resulting equation and make the hypothesis of homogeneity, i.e. $< a_i a^*_j >= N_i \delta(k_i - k_j)$. The resulting equation is the following:

$$\frac{\partial N_0}{\partial t} - 2IM\left[\int U^{(1)}_{0,1,2} < a^*_0 a_1 a_2 > \delta_{0-1-2} dk_{12} + \int 2U^{(1)}_{2,1,0} < a^*_0 a^*_1 a_2 > \delta_{01-2} dk_{12} + \int U^{(3)}_{0,1,2} < a^*_1 a^*_2 a^*_0 > \delta_{012} dk_{12}\right] = 0 \qquad (15)$$

where IM stands for imaginary part. We have introduced three third order correlators of the form $< a_0 a_1 a_2 >= J_{123}\delta(k_0 + k_1 + k_2)$ for which we need to write three equations. We will just discuss the first of them, i.e. $< a^*_0 a_1 a_2 >$. We need to derive an evolution equation for J_{1-2-3}. In order to do that we first take the time derivative of $< a_1 a_2 a^*_0 >$ and by making use of (3) we obtain the following equation:

$$\left[\frac{\partial}{\partial t} + i(\omega_1 + \omega_2 - \omega_0)\right] J_{0-1-2}$$
$$-i\int U^{(1)}_{0,p,q} < a^*_p a^*_q a_1 a_2 > \delta_{12-p-q} dk_{pq}$$
$$+i\int U^{(1)}_{q,p,2} < a_1 a^*_0 a^*_p a_q > \delta_{0-1p-q} dk_{pq}$$
$$+i\int U^{(1)}_{q,p,1} < a_2 a^*_0 a^*_p a_q > \delta_{0-2p-q} dk_{pq} = 0. \qquad (16)$$

Note that a number of fourth-order correlators of the form of $< a_1 a_2 a_3 a_4 >$ have been introduced; in order to avoid deriving new equations for those objects a clousure is needed. We adopt the quasi-gaussian approximation, i.e. we write the fourth order correlator as the sum of the product of second order correlators. For example:

$$< a_p a_q a^*_1 a^*_2 > \simeq N_1 N_2[\delta(k_p - k_1)\delta(k_q - k_2) + \delta(k_p - k_2)\delta(k_q - k_1)], \qquad (17)$$

Note that the use of this approximation for three-wave interactions is a priori less intuitive with respect to the one applied to the wave four-wave interaction equation. This is mainly due to the fact that, while the starting equation for the four-wave interactions consists of free waves, it is well known in the present case the Stokes contribution introduces a deviation from gaussianity in the data, therefore this hypothesis should be carefully checked numerically. Introducing this approximation in (16) and using the properties of the delta function, we obtain the following equations:

$$\left[\frac{\partial}{\partial t} + i\Delta^{(1)}\omega\right] J_{0-1-2} = 2iU^{(1)}_{0,1,2}[N_1 N_2 - N_1 N_0 - N_2 N_0] \qquad (18)$$

where $\Delta^{(1)}\omega = \omega_1 + \omega_2 - \omega_0$. Similar equations can be obtained for J_{0+1-2} and J_{0+1+2}. Those equations with (15) provide our system of equations that describe the dynamics of an ensemble of waves. Equation (18) can be easily written in the form

$$\frac{\partial J'_{0-1-2}}{\partial t} = 2iU^{(1)}_{0,1,2}[N_1N_2 - N_1N_0 - N_2N_0]e^{i\Delta^{(1)}\omega t} \quad (19)$$

where $J_{0-1-2} = J'_{0-1-2}e^{-i\Delta^{(1)}\omega t}$. We integrate equation (19) in time from 0 to t supposing that at $t = 0$, $J'_{0-1-2} = 0$; moreover we make the hypothesis that the function $N_1N_2 - N_1N_0 - N_2N_0$ changes on a time scale slower than the function $e^{-i\Delta^{(1)}\omega t}$, therefore we obtain:

$$J'_{0-1-2} = i2U^{(1)}_{0,1,2}[N_1N_2 - N_1N_0 - N_2N_0]\int_0^t e^{i\Delta\omega\tau}d\tau. \quad (20)$$

The same arguments can be applied to the other equations for the third-order correlator. Putting together all the results, the kinetic equation can finally be written in the following form:

$$\begin{aligned}\frac{\partial N_0}{\partial t} = \int 4|U^{(1)}_{0,1,2}|^2[N_1N_2 - N_1N_0 + \\ -N_2N_0]\delta_{12-0}\frac{sin(\Delta^{(1)}\omega t)}{\Delta^{(1)}\omega}dk_{12} - \\ \int 8|U^{(1)}_{2,1,0}|^2[N_1N_0 - N_1N_2 + \\ -N_0N_2]\delta_{10-2}\frac{sin(\Delta^{(2)}\omega t)}{\Delta^{(2)}\omega}dk_{12} + \\ \int 4|U^{(3)}_{0,1,2}|^2[N_1N_2 + N_1N_0 + \\ +N_2N_0]\delta_{120}\frac{sin(\Delta^{(3)}\omega t)}{\Delta^{(3)}\omega}dk_{12}\end{aligned} \quad (21)$$

where $\Delta^{(1)}\omega = \omega_1+\omega_2-\omega_0$ and $\Delta^{(2)}\omega = \omega_2-\omega_1-\omega_0$ and $\Delta^{(3)}\omega = -\omega_2-\omega_1-\omega_0$. This equation represents an evolution equation for the wave action to be applied in the shallow water regime. Note that the equation appears quite complicated also for numerical simulations and therefore we will here derive an evolution equation starting directly from the Korteweg - de Vries. The derivation is analogous to the one just outlined with the only difference that here the evolution can be directly written for the energy wave spectrum $< \eta_{k_i}\eta^*_{k_j} >= P_{k_i}\delta(k_i - k_j)$ and not for the wave action spectrum N_k. The resulting equation has the following form:

$$\begin{aligned}\frac{\partial P_0}{\partial t} = \alpha^2 k_0 \int \frac{sin(\Delta\omega t)}{\Delta\omega}[k_0P_1P_2 + \\ -k_1P_0P_2 - k_2P_0P_1]\delta(k_1 + k_2 - k_0)dk_{12},\end{aligned} \quad (22)$$

where $\Delta = \omega_1 + \omega_2 - \omega_0$ with $\omega_i = c_0k_i - \beta k_i^3$. α and β are just the two coefficients that appear respectively in front of the nonlinear and dispersive term in the KdV equation (see equation (14)). From the KdV equation it is also possible to estimate some statistical properties of the surface elevation such as the skewness defined as:

$$s = \frac{< \eta^3 >}{< \eta^2 >)^{3/2}} \quad (23)$$

We will now find a relation between this parameter and the wave spectrum for an homogeneous wave field. After some calculations it can be shown that the skewness takes the following form:

$$\begin{aligned}s = \frac{3}{2}\frac{c_0}{h}\Big(\int P_1dk_1\Big)^{-3/2}\int[k_1P_2P_3 - k_2P_1P_3 + \\ -k_3P_1P_2]\delta_{1-2-3}(1 - Cos(\Delta\omega t))/\Delta\omega)dk_{123}\end{aligned} \quad (24)$$

In order to get some feeling for the physical consequences of equation (24) we consider a wave spectrum of the form $P(k) = P'(k)\delta(k - k_0)$ and obtain the simple relation:

$$\begin{aligned}s = \frac{3}{2}\frac{k_0\sqrt{P(k_0)}}{(k_0h)^3}(1 - Cos(c_0k_0^3h^2t))) \sim \\ \sim Ur(1 - Cos(c_0k_0^3h^2t)),\end{aligned} \quad (25)$$

where primes have been omitted. Therefore the skewness is proportional to the Ursell number. An interesting result that can be intuitively seen from equation (26) is that if at time $t = 0$ a narrow banded spectrum with random phases propagates on a flat bottom, the skewness will oscillate on a time scale of the order of $1/(c_0k_0^3h^2)$. This result is consistent with the well know Fermi Pasta Ulam recurrence phenomenon typical of integrable system such as the KdV equation. Note that this result applies only in the case of flat bottom; if a constant slope is present we expect that the skewness would increase until breaking takes place.

IV. CONCLUSIONS

In this paper we have discussed a number of issues concerning the propagation of waves in shallow water. We have shown that the canonical transformation usually adopted to build the kinetic equation has some serious limitations when waves propagate into shallow water. We therefore have derived from the most general equation describing three-wave interactions a kinetic equation that describes the quasi-resonant interactions between these waves. The methodology is applied also to the Korteweg -de Vries equation from which we were able to derive a simple formula for the skewness that shows recurrence behaviour just as the deterministic equation does. The next step (already part of the on going research) is to test the validity of the derived kinetic equation with Montecarlo simulation of the deterministic equation.

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