# ON THE ROLE OF SELF-SIMILAR SOLUTIONS IN THE EVOLUTION OF WIND-DRIVEN OCEAN WAVES

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## Theoretical background

The Hasselmann kinetic equation [1] is a basis of many models of wind-wave forecasting. It describes changes of wave action (or wave energy) spectral density $N_{\boldsymbol{k}}$ due to effects of generation by wind, dissipation and four-wave resonant interactions

$$\partial N_{\mathbf{k}} / \partial t + \nabla_{\mathbf{k}} \omega \nabla_{\mathbf{x}} N = S_{nl} + S_{in} + S_{diss} \qquad (1)$$

So far knowledge of wind wave generation and dissipation is rather short and is based mainly on empirical parameterizations of $S_{in}$ and $S_{diss}$ in (1). At the same time, the nonlinear transfer term – collision integral $S_{nl}$ is known from “the first principles”. The basic results of solution of Eq.1 are limited by stationary solutions of the conservative kinetic equation:

$$S_{nl} = 0 \qquad (2)$$

The Rayleigh-Jeans solutions describe local balance of each resonant quadruplet of water wave harmonics, while solutions of other type, the so-called Kolmogorov-Zakharov (KZ) solutions correspond to a dynamical equilibrium when input and output for each element of the nonlinear system are balanced, i.e. spectral fluxes of integrals of motion are constant. Two solutions of this type play a fundamental role – direct cascade solution with constant flux of energy from large to small scales [3] and inverse cascade solution that describes constant flux of wave action to large scales [4].

In this paper we present theoretical and numerical analysis of families of self-similar solutions of the kinetic equation for water waves. These solutions can be considered as a generalization of the KZ solutions that describe adequately features of real wind wave spectra: pronounced peakedness, anisotropy, downshifting of developing wave spectra *etc*. Additionally, these solutions can be related quite naturally to conventional experimental parameterizations of the spectra [2] that imply self-similarity, or, more, universality of the spectra. The main point of the present study is: ***nonlinear*** *transfer is a key physical mechanism of evolution of wind-wave spectra*. Accepting this hypothesis and using an important feature of homogeneity of collision integral

$$S_{nl} \sim N^3 \mathbf{k}^{19/2} \qquad (3)$$

($\mathbf{k}$ is the wave vector), one can obtain self-similar solutions for the "conservative" kinetic equation (1)

$$dN_{\mathbf{k}} / dt = S_{nl} \quad (4)$$

as approximate ones for Eq.1. In the so-called case of duration-limited growth of wind waves (fetch limited-growth and the corresponding stationary inhomogeneous solutions can be considered in a similar way) the non-stationary homogeneous solutions of (1) take the form

$$N_{\mathbf{k}} = a t^{\alpha} U_{\beta}(b \mathbf{k} t^{\beta}) \quad (5)$$

Parameters of the solutions (5) obey

$$a = b^{19/4}; \quad \alpha = (19\beta - 2)/4 \quad (6)$$

Parameters $a$ and $b$ are determined by initial conditions while for $\alpha$ and $\beta$ the effect of formally small wind input and wave dissipation in (1) should be taken into account. The corresponding condition can be formulated in the form of balance equation

$$d\langle N_{\mathbf{k}} \rangle / dt = \langle S_{in} + S_{diss} \rangle \quad (7)$$

for total wave action, input and dissipation (here brackets <...> mean integration over the wave vector space). Within (7) parameters $\alpha$ and $\beta$ can be specified as functions of the exponent $r$ of power-like growth of total wave action

$$\int N_{\mathbf{k}} d\mathbf{k} \sim t^r; \quad \alpha = (19r + 4)/11; \quad \beta = (4r + 2)/11 \quad (8)$$

Nonlinear transfer and growth of total wave action (energy) appear to be split in the model (4, 7). It implies that forms of wind-driven spectra are determined, first of all, by features of resonant wave-wave interactions while magnitudes of these spectra depend on total wind input mainly. This physical model has been tested in an extensive numerical study in order to show its consistency with numerical solutions of full kinetic equation (1) and with existent spectral models of wind-wave evolution.

**Numerical results**

Numerical approach was aimed, first of all, at justification of theoretical analysis of self-similar solutions of the Hasselmann equation (1). Details of the numerical algorithm used in this paper are published in [7, 8]. Preliminary results of the extensive numerical studies showed adequate accuracy and stability of calculations in a wide range of parameters of wave field and wind input [9, 10].

Two groups of numerical experiments have been performed to detail the properties of self-similar solutions and their relevance to real wind-wave spectra. "Academic" runs with artificial functions of wave input $S_{nl}$ were designed to obtain asymptotic (at large time) solutions in a wide range of indexes of self-similarity $r$ (or $\alpha, \beta$) as a reference for further numerical experiments with "real" wave generation. Perfect coincidence of the resulting solutions with predictions of self-similarity analysis was found. The forms of the solutions depend on the parameters of self-similarity very slightly, i.e. the numerical wind wave spectra have quasi-universal forms. It is consistent with universal forms of experimental parameterizations of wind-wave spectra [2, 6].

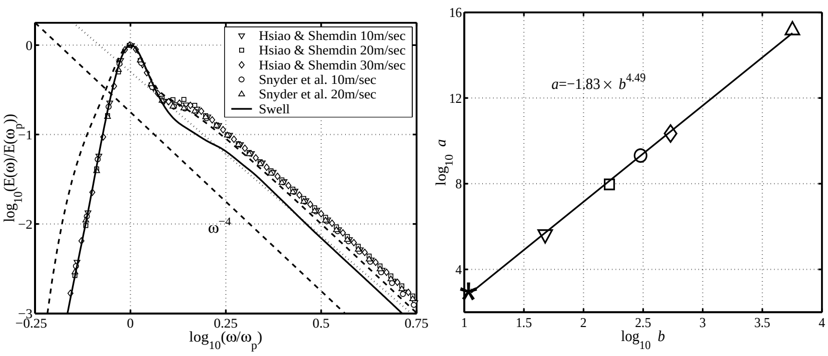


Fig.1. Left panel – non-dimensional frequency spectra $E(\omega)/E(\omega_p)$ as functions of non-dimensional wave frequency for different wave inputs (in legend). The JONSWAP spectrum for the standard peakedness $\gamma$=3.3 is shown by dashed curve. Right panel - the dependence of the solutions parameters $a$ and $b$ (6), swell scaling is given by ★.

Experiments with conventional functions of wave input [12-14] showed rather good agreement of the numerical and theoretical results. Forms of the solutions for different cases of wave input are found to be very close to each other and to solutions of the "academic" series. This universality feature is demonstrated by Fig.1 where normalized frequency spectra are shown for different input terms $S_{in}$, for swell solution ($S_{in}=0$) and for JONSWAP spectrum with the standard set of parameters [6]. An additional argument for this property – a universal scaling of parameters $a$ and $b$ (6) – is illustrated by right panel in Fig.1. It should be stressed that these parameters have been specified by positions and magnitudes of the peaks of the solutions. In other words, the self-similarity features are evidently more pronounced for high magnitudes of solutions where evolution is governed by nonlinear transfer term

$S_{nl}$ mainly, while in the solutions periphery this evolution can be essentially non-self-similar.

The correct definition of the self-similarity parameters is required in the case of "real" inputs when a non-self-similar background co-exists with a self-similar "core" of the solution. The background can contaminate significantly the self-similarity features. It is seen in behavior of parameters of wind wave growth (see Fig.2): $p$ – exponent of total energy growth and $q$ – exponent of mean frequency downshift. Left panel of Fig.2 shows the directly calculated $p$ and $q$, while in the right panel similar quantities are calculated as functions of parameters of peaks of solutions $\alpha$ and $\beta$ (see Eq. 8). One can see that the last definition of the parameters ($p_{exp}$, $q_{exp}$) fits theoretical dependencies better.

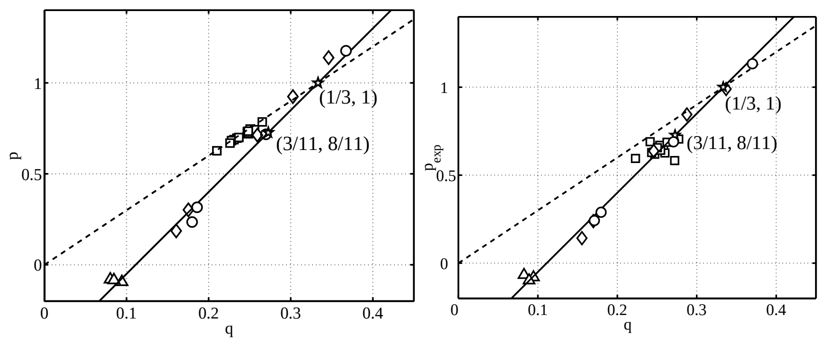


Fig.2. Left panel – exponents $p$ and $q$ for power-like approximations of total energy and mean frequency of the kinetic equation solutions. Right panel - exponents $p_{exp}$ and $q_{exp}$ calculated for parameters $\alpha$ and $\beta$ of the solution peak. O - isotropic "academic" runs; ◇ - anisotropic "academic" runs; △ - swell; □ - "real" wave pumping. Exponents for constant wave action and wave energy inputs are given by ★. Hard line shows theoretical dependence of $p$ on $q$, dashed line corresponds to the Toba law.

The dominating role of nonlinear transfer is justified by asymptotic behavior of the nonlinear transfer term $S_{nl}$ and fluxes of wave action, energy and momentum. The "conservative" kinetic equation (4) being rewritten in self-similar variables $\boldsymbol{\xi}=\mathbf{k}t^{\beta}$ gives the remarkable feature of self-similar behavior: nonlinear transfer term can be calculated explicitly as a linear function of $U_{\beta}(\boldsymbol{\xi})$

$$\alpha U_{\beta} + \beta\boldsymbol{\xi}\nabla_{\xi} U_{\beta} = S_{nl}(U_{\beta}(\boldsymbol{\xi})) \qquad (9)$$

Thus, $S_{nl}$ and all fluxes can be calculated in primitive variables by simple rescaling in time. One can show that signs of fluxes for the domain of validity of self-similar regimes are fixed and correspond to inverse cascading of wave action, energy and momentum. Comparison of directly calculated $S_{nl}$ and fluxes with their asymptotic counterparts determined by (9) are shown in fig.3 for two different times. Fig.3 shows strong tendency to self-similar behavior near peaks of solutions. Solution and nonlinear transfer term $S_{nl}$ are growing with time, while total wave input is tending to be constant. It is consistent with our starting point on dominating role of nonlinear transfer. Wave action flux in Fig.3 varies slightly with time, while fluxes of wave energy and momentum decay with time, as it is predicted by self-similar asymptotic (5). Such behavior can be related naturally to the KZ power-like spectra with slowly varying in time parameters.

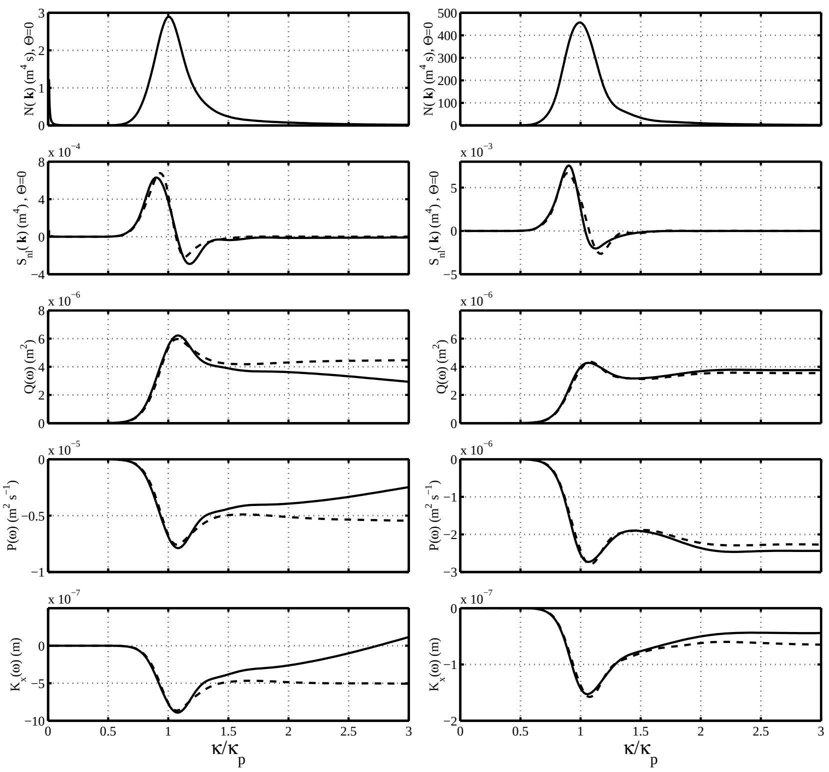


Fig.3. Direct calculation of $S_{nl}$ and fluxes vs their asymptotical profiles predicted by Eq.9 for wave input [12], wind speed 10 m/sec. Plots age given for dimensional values and non-dimensional wave number $\kappa/\kappa_p$. Left – time 4.15 hours; right – time 65 hours.

### Discussion

Experimental parameterizations of wind-wave spectra are based on similarity analysis proposed by Kitaigorodskii [15] and a concept of self-similarity and universality of observed wind-wave spectra [2]. Thus, the experimentally measured parameters can be related straightforwardly with the corresponding theoretical and numerical results. Formally, the JONSWAP parameterizations [2, 6] are valid for the fetch-limited wave growth, while our numerical results are for duration limited case only. Thus, the perfect coincidence of numerical and experimental wave spectra found in the study can be considered as a justification of the model (4, 7). The "conservative" part (4) of the model describes universal forms of spectral distributions while balance equation (7) determines rates of spatio-temporal evolution of the spectra. Universality of "form functions" $U_\beta(\xi)$ (their weak dependence on self-similarity indexes) opens good prospects for wave forecasting based on the model (4, 7) .

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