THE EXCEEDENCE PROBABILITY OF WAVE CRESTS CALCULATED BY THE SPECTRAL RESPONSE SURFACE METHOD

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1.0 INTRODUCTION

The prediction of extreme crest elevations is fundamental to the design of marine structures, irrespective of whether they are intended for deep-water offshore locations or shallow-water coastal locations. In undertaking design extreme crest elevations, specified in terms of either a 1 in 100 year or 1 in 10,000 year event, are relevant to both the overall geometry of the structure, particularly the setting of deck elevations, and the sizing of individual members necessary to support the applied loads. For example, in the design of a fixed structure deck elevations are traditionally set to maintain an effective air-gap, thereby preventing the impact of even the largest wave crests on the underside of the deck structure. In contrast, the wave-induced forcing acting on the legs of the structure cannot be avoided and must be carefully assessed. If the relevant Keulegan-Carpentar number *(KC)* is large,

$$KC = \frac{\hat{U}T}{D} > 20 \tag{1.1}$$

where $\hat{U}$ is the amplitude of the horizontal wave-induced orbital velocity, T is the wave period and D is the diameter of the cylindiical leg, the forcing will be drag dominated. In this case, the forcing is dependant upon the square of the wave-induced orbital velocity. As a result, the maximum horizontal forcing, contributing to both the maximum base shear and the maximum over-turning moment, occurs directly beneath the largest wave crests and its magnitude increases with the square of the wave amplitude.

Other examples where the assessment of extreme crest elevations is critical to the design process include the design of floating structures, particularly the determination of the applied loads, the corresponding vessel response and the occurrence of green water inundation. Likewise, crest elevations are also key to the design of coastal structures for shoreline protection and flood prevention. In these cases some degree of over-topping may be permissible, but its estimation must be based upon a clear understanding of extreme crest height distributions.

Given the practical importance of extreme crest height distribution, the starting point for this paper lies in two recent advances. First, the development of methods based on spectral response surfaces (Tromans & Suastika (1998)). When this approach is combined with a first order reliability method (FORM) it allows the rapid determination of design criteria associated with the occurrence of extreme events. Secondly, the recent development of new wave models allowing an exact (or fully nonlinear) description of the evolution of large waves within realistic sea states that are broad-banded in terms of their frequency spectrum and directionally spread. The purpose of the present paper is to combine these approaches in an attempt to determine whether the distribution of extreme crest heights is significantly affected by the nonlinearity of the local wave field and to investigate the role played by directionality.

The paper commences in §2 with a brief description of the spectral response surface method, its application to the prediction of extreme crest heights and its relation to the NewWave model (Tromans et al (1991)). Section 3 outlines recent developments in the descriptions of extreme waves, explaining how recent models may be applied to provide fully nonlinear calculations of extreme transient waves that are broad-banded in both frequency and direction. Section 4 combines the methods outlined in §2 and §3 compares a number of results in which crest elevations with a given probability of exceedence are calculated using a linear, second-order and a fully nonlinear wave model. Although some aspects of these results are preliminary, new results are presented and discussed and the significance of nonlinearity in determining crest heights clearly identified. The paper concludes in §5 with a summary of the calculations made to date, an assessment of their practical significance and a discussion of additional work that is both on-going and planned.

2.0 SPECTRAL RESPONSE SURFACES AND CREST HEIGHT ESTIMATIONS

In broad terms the spectral response surface (SRS) method involves the application of conventional first order second moment reliability methods (FORM), Madsen et al. (1986) and Melchers (1987), to problems involving a spectral representation. The method has previously been applied to several important issues in ocean engineering including extreme loads on space frame structures (Tromans & Van Dam (1996)), the 'ringing' or dynamic response of offshore structures (Tromans & Suastika (1998)), the shape and history of extreme crest elevations (Tromans & Taylor (1998)) and, more recently, second-order estimates of crest statistics (Tromans & Vanderschuren (2002)).

In the latter case the method effectively involves random directional wave modelling in the probability domain. This has two distinct advantages. First, the SRS method enables a number of linear theories for the distribution of maxima and the structure of a process in the vicinity of these maxima (Lindgren (1970), Tromans et al. (1991) & Phillips et al. (1993)) to be generalised for problems involving nonlinearity. Secondly, it avoids the need for long time-domain simulations. This becomes particularly important with increasing nonlinearity. If a sea state is assumed linear, time-domain simulations are easily undertaken, but largely irrelevant given the theories noted above. If it is assumed to be weakly nonlinear, and modelled using second-order theory (Sharma & Dean (1981)), time-domain simulations are straightforward (Forristall, 1998) but computationally intensive.

With increased nonlinearity, including third and higher-order effects, long time-domain simulations are simply not possible. Since the purpose of the present paper is to investigate the influence of fully nonlinear wave calculations on crest statistics, the SRS method is an obvious choice.

Adopting a linear representation, the ocean surface elevation may be defined by the sum of a large number (N) of random, narrow-banded, frequency components, each of which is normally distributed. All of the components are assumed independent and uncorrelated. If a_j is the random amplitude of the j^{th} component ($l \le j \le N$) ω_j its frequency, $\mathbf{k_j}$ the wave number vector $\left(k_{xj}, k_{yj}\right)$, q_j the random phase angle and t the time, the water surface elevation may be written as:

$$h^{(1)}(t) = \sum_{j=1}^{N} h_j = \sum_{j=1}^{N} a_j \; Cos\,(q_j + \boldsymbol{k_j}\,\boldsymbol{x} - w_j\,t) \qquad (2.1)$$

Where the superscript $^{(1)}$ indicates the first order or linear approximation and $\boldsymbol{x}$ defines the horizontal spatial location.

Following earlier work the frequency components can be transformed onto standardised (unit variance, zero mean) variables by dividing each by its standard deviation.

$$x_j = \frac{h_j}{s_j} \quad and \quad \tilde{x}_j = \frac{\tilde{h}_j}{\tilde{s}_j} \qquad (2.2)$$

where the superscript $\tilde{}$ indicates a Hilbert transform. In the present context the Hilbert transform of a variable may be thought of as the value of the variable with its phase shifted by $\pi/2$. Consequently, $\tilde{s}_j = s_j$. Combining equations (2.1) and (2.2) it is easily shown that

$$a_j = \left(\left(s_j x_j\right)^2 + \left(\tilde{s}_j \tilde{x}_j\right)^2\right)^{\frac{1}{2}} \qquad (2.3)$$

and

$$q_j = \tan^{-1}\left(\frac{x_j}{\tilde{x}_j}\right) \qquad (2.4)$$

Accordingly, it is clear that if a FORM is applied to provide a solution in terms of the standardised variables (x_j, $\tilde{x}_j$) the component amplitudes and their phasing can be defined. It therefore follows that the local water surface profile and the crest elevation are also defined. Recalling that the joint density function of the standardised variables $\left(x_j\,,\ \tilde{x}_j\right)$is a unit variance normal and that the variables are uncorrelated, the surfaces of constant probability density are concentric spheres in the space of the standarised variables. The probability density is highest at the origin and falls monotonically as a function of distance from the origin.

In a typical FORM analysis one might identify a surface of constant response (in our case a given crest elevation) and would then seek to identify the point on the surface that lies closest to the origin. This is denoted as the design point $\left(x_j^*, \tilde{x}_j^*\right)$and its distance from the origin gives the probability that the specified response will occur. In the present case we adopt a different approach in which we search along a given hyper-sphere of constant probability in order to find the maximum response for a given probability of exceedence. Although different, the design point is located using established procedures based on Lagrange's method of undetermined multipliers (Melchers, 1987).

In earlier related work Tromans & Vanderschuren (2002) have adopted exactly this approach and have demonstrated that the SRS method is in very good agreement with statistical distributions based on a time-domain analysis for exceedence probabilities from 0.5 down to $5x10^{-3}$. For values less than $5x10^{-3}$ small differences arise but they may be accounted for in terms of the scarcity of data relating to the description of extreme crests in the time-domain simulation. They also implemented the second-order description of the water surface elevation due to Sharma & Dean (1981) and showed that when this is optimised using the SRS method the resulting crest elevations are in very good agreement with the simple process of applying a second-order correction to the linear NewWave (Tromans et al (1991)).

3.0 RECENT ADVANCES IN WAVE MODELLING

It is well established that large ocean waves do not arise as part of a regular wave train, but occur as isolated and transient events within a random or irregular sea, which is broad-banded both in its frequency and its directional distribution. In order to describe the evolution of such waves it has been shown that an appropriate wave model must be unsteady (or capable of incorporating the underlying frequency distribution), fully nonlinear (since we are primarily concerned with the largest waves), and able to incorporate the directional spread. Until very recently (Bateman, Swan & Taylor (2001)) it has not been possible to incorporate these three key requirements. Indeed, in terms of engineering design most of the commonly applied wave models either incorporate the nonlinearity (eg fifth-order Stokes model (Fenton, 1985)) or the unsteadiness (eg linear random wave theory) and do little in regard of the directionally.

The motivation behind the development of this new class of wave model serves as an excellent of the interplay between laboratory observations, field data and numerical calculations. The initial interest in the nonlinearity of these waves and, in particular, its relationship to directionality, arose as a result of comparisons between an analysis of field observations reported by Jonathan & Taylor (1996) and laboratory observations reported by Baldock, Swan & Taylor (1996). In this latter study data defining the water surface elevation corresponding to a number of uni-directional wave groups were reported and comparisons with linear theory suggest that the waves may become very nonlinear, with maximum crest elevations as much as 40% larger than linear theory and 30% larger than second-order theory. In contrast, the field observations suggest that for the waves observed the crest elevations were reasonably well predicted by second-order theory, indicating only weak nonlinearity.

An obvious explanation for this difference lies in the directionality of real ocean waves. Accordingly, Johannessen & Swan (2001) undertook a second and very detailed laboratory study in which focussed waves were generated with varying directional spread ranging from uni-directional cases (confirming the earlier findings of Baldock, Swan & Taylor (1996)) to very short-crested waves involving a large directional spread (s=7, where s is the Mitsuyasu spreading parameter). In this and much of the subsequent discussion the notion of a 'focussed' wave merely defines one in which the phasing of the freely propagating wave components and/or their direction of propagation is pre-determined so as to achieve the summation of wave crests at one point in space and time. This represents the most effective method of achieving a large isolated wave crest within a broad-banded sea state and is believed to be representative of the evolution of large waves in the field. Indeed, this concept of focusing lies at the heart of the (linear) NewWave Model that has been successfully compared to a number of field observations (Razario et al (1990)).

The data presented by Johannessen & Swan (2001) confirm that the directionality is very significant in relation to the overall nonlinearity of the largest waves and offers a possible explanation for the difference between uni-directional laboratory data and field observations. Indeed, the influence of directionality was so significant that the results cast considerable doubt on the desirability of undertaking model studies using uni-directional waves.

To further examine the influence of directionality Johannessen & Swan (2002) sought to investigate the laboratory data further. To achieve this they required both temporal and spatial descriptions of the water surface elevation, $h(t)$ and $h(x,y)$ respectively, where (x,y) define a horizontal plane located at the still water level and *t* the time. Although the former description can be provided by laboratory observations, the latter cannot or, at least, not with sufficient resolution. Accordingly, Johannessen & Swan (2002) adopted a three-dimensional extension of an exact uni-directional wave model proposed by Fenton & Rienecker (1982). This model is based upon an approach first proposed by Longuet-Higgins & Cokelet (1976). If the wave motion is assumed irrotational, the nonlinear free surface boundary conditions can be expressed as:

$$h_t = f_z - h_x f_x - h_y f_y \tag{3.1}$$

$$f_t = -gh - \frac{1}{2}\left|\nabla f\right|^2 \tag{3.2}$$

Where z is measured from the still water level upwards, g is the gravitational acceleration and f is the velocity potential defined so that the velocity components in the (x, y, z) directions are given by (u, v, w) $= \nabla f = (\partial_x, \partial_y, \partial_z) f$. These conditions respectively define the kinematic and dynamic free surface boundary conditions, which require the fluid at the free surface to remain there and the pressure on the water surface to be constant. Using these equations (or their uni-directional equivalent) Longuet-Higgins and Cokelet noted that if one had a spatial description of the wave field at some initial time $t = t_0$, $h(x, y, t_0)$ and $f(x, y, z = h_0, t_0)$, equations (3.1) & (3.2) can be used to time-march the wavefield to all subsequent times.

Expanding the method of Fenton and Rienecker (1982), Johannessen & Swan (2002) achieved extremely good description of the evolution of the laboratory-scale wave field. Unfortunately, the computational efficiency of this method in such that it cannot be readily applied to problems involving the very large range of length-scales appropriate to realistic ocean spectra. This problem has now been overcome by Bateman, Swan & Taylor (2001 & 2002). In this series of papers they adopted the highly efficient computational procedure proposed by Craig & Sulum (1993) and again extended the method to apply to directionally spread waves. The overriding advantage of this technique is that by applying a Taylor series expansion of the Dirichlet-Neumann operator the solution can be formulated in terms of the surface parameters alone. This dimensional reduction yields considerable computational savings, not least because it avoids the need for the large matrix inversion on which both the Fenton & Rienecker (1982) and the Johannessen & Swan (2002) models rely. As a result, the model proposed by Bateman, Swan & Taylor (2001), and its subsequent extension to include the calculation of the water particle kinematics (Bateman, Swan & Taylor 2002), allow the evolution of realistic broad-banded spectra (in both frequency and direction) to be calculated with high accuracy.

This wave model represents the state-of-the-art in terms of wave predictions and lies at the heart of the present study. To apply the model one simply needs to identify the underlying frequency and directional spectra and to choose an initial time, $t = t_0$, at which the wave field is fully dispersed. With no large waves present within the computational domain, a linear solution can be used to specify the initial conditions, $h(x, y, t_0)$ and

$f\left(x, y, z = h, t_0\right)$, and the solution time-marched to provide a fully-nonlinear description of an extreme wave event that is, to all intents and purposes, exact.

4.0 DISCUSSION OF RESULTS

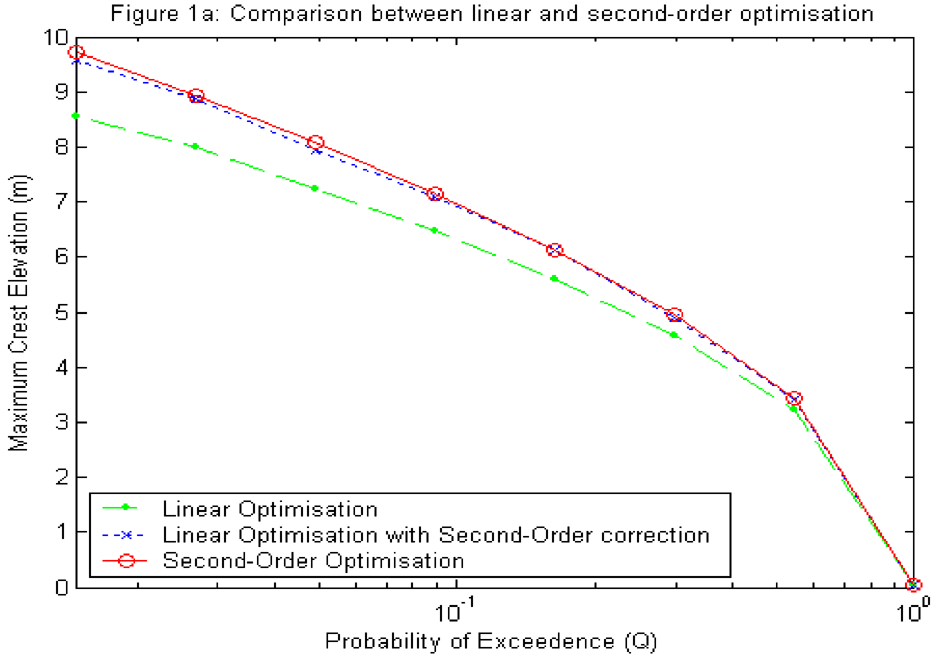
The results presented in this section correspond to a JONSWAP spectrum in deep water with a peak period of T_p = 12.8s, a significant wave height of H_s = 12.0m and a peak enhancement factor of g = 2.3. Initial calculations for this and several other cases confirm that if the water surface elevation is described by a linear model the exceedence probabilities of crest elevations calculated using the SRS method exactly reproduce the values given by the Rayleigh distribution. Likewise, the water surface elevations, ?(t), corresponding to these extremes are identical to the corresponding NewWave profiles (Tromans et al (1991)).

Having reproduced the established linear behaviour, a second set of calculations sort to identify the difference between:

(a) Applying the SRS method to a second-order description of the water surface elevation based on the analytical solution proposed by Sharma and Dean (1981). This provides a linear input spectrum that has been optimised to take account of the second-order frequency-sum and frequency-difference terms.
(b) Applying the SRS method to a linear description of the water surface elevation and having identified the crest elevation (and the corresponding wave profile) for a given exceedence probability, applying a second-order correction again using the solution due to Sharma and Dean (1981). The principle difference being that in this case the linear input spectrum has been optimised on the assumption that the sea surface is linear.

The results of this process are given on Figures 1a and 1b which respectively contrast the water surface elevation, ?(t), and the underlying frequency spectrum, a(?). The difference between the results is small and appears to be consistent with the earlier findings of Tromans & Vanderschuren (2002).

Having established the success of the approximate method outlined in (b) above, a similar approach is adopted in respect of the fully nonlinear numerical calculations. Although this approach cannot, as yet, be fully justified the results do, at least, give some guidance as to the significance of the fully nonlinear interactions. At this point it is perhaps relevant to note that work is presently on-going to determine a linear input spectrum which has been optimised to take account of the full nonlinearity. Unfortunately, this task is extremely time consuming since it involves repeated runs of the fully nonlinear model at each stage of the optimisation process. However, at this stage our preliminary results suggest that there is little difference in the extreme values produced, particularly as the directionality increases,



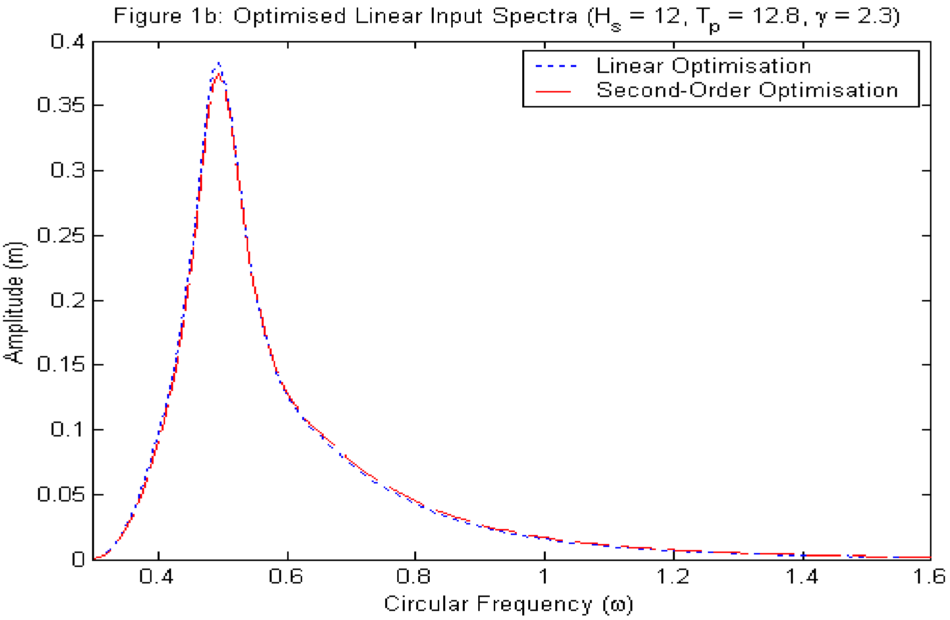


Figure 2 presents the results arising from the linearly optimised solution (method (b) above) and contrasts the crest elevation for a given probability of exceedence for a linear, a second-order and a fully nonlinear simulation. These results relate to a uni-directional wave field and suggest that in the case of the largest waves the contribution from the third and higher-order terms may be at least as large as those arising at second order.

Confirmation of this is given in Figure 3, which contrasts the wave profiles corresponding to the second-order model and those from the fully nonlinear calculations.

Figure 2: Maximum crest elevations for various probabilities of exceedance in uni-directional wavefield

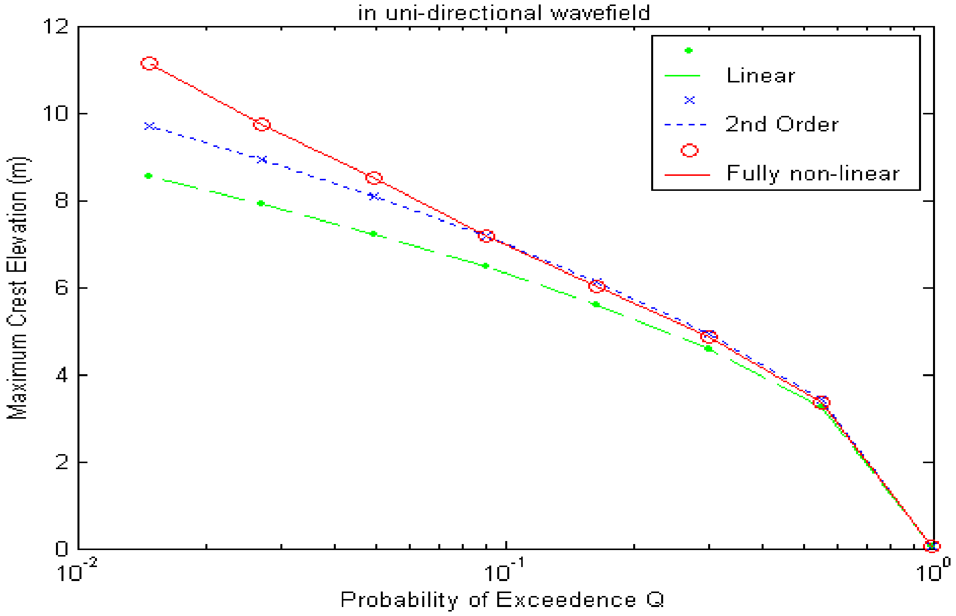
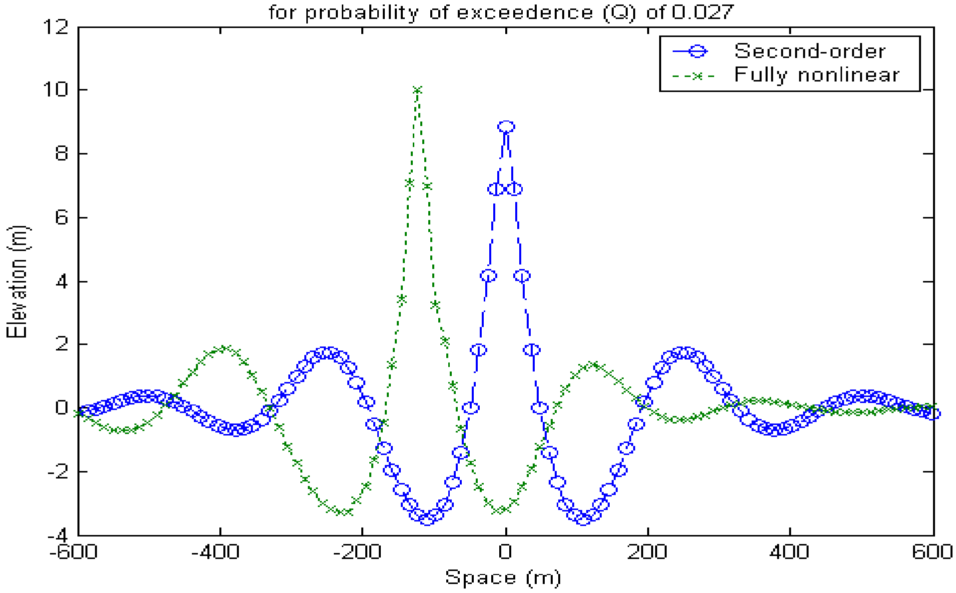


Figure 3: Comparison between fully nonlinear and second-order surface profiles for probability of exceedence (Q) of 0.027



The wave profiles corresponding to three of the fully nonlinear cases are presented on Figure 4a. In the largest wave cases it is clear that the crest-trough asymmetry becomes very marked. This is consistent with the earlier experimental observations reported by Baldock, Swan and Taylor (1996) and is in part explained by the changes in the wave spectra noted on Figure 4b. As the wave height increases there is a marked transfer of energy to the higher frequency components, allowing larger crest elevations to evolve.

Figure 4a: Surface profiles of three waves with varying probability of exceedence (Q) at time of greatest crest height.

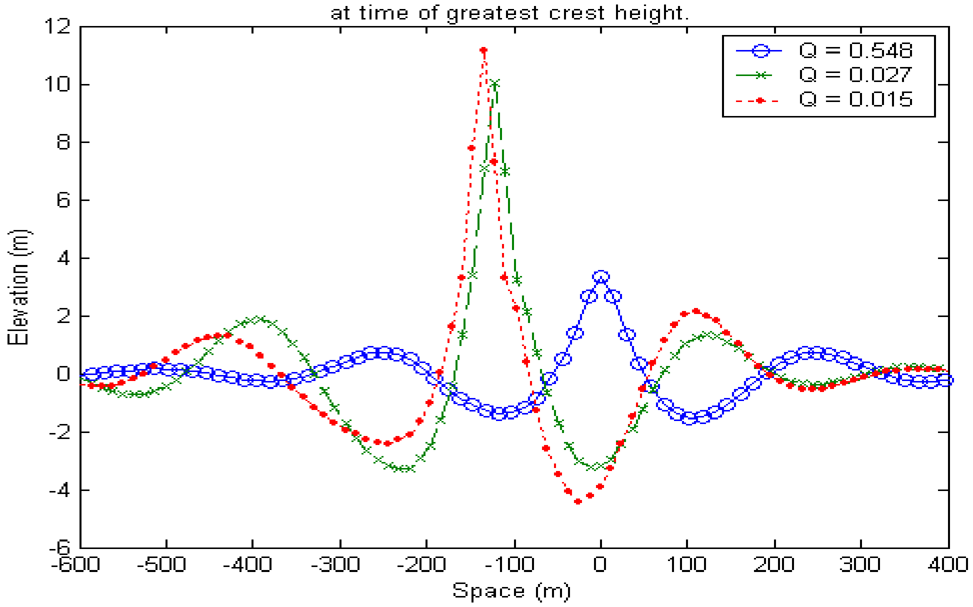
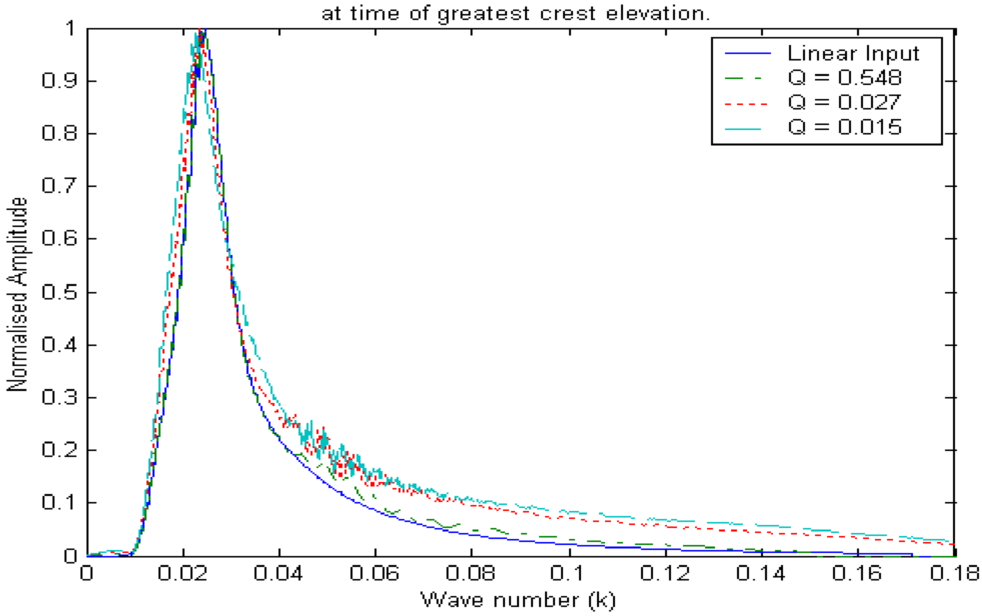
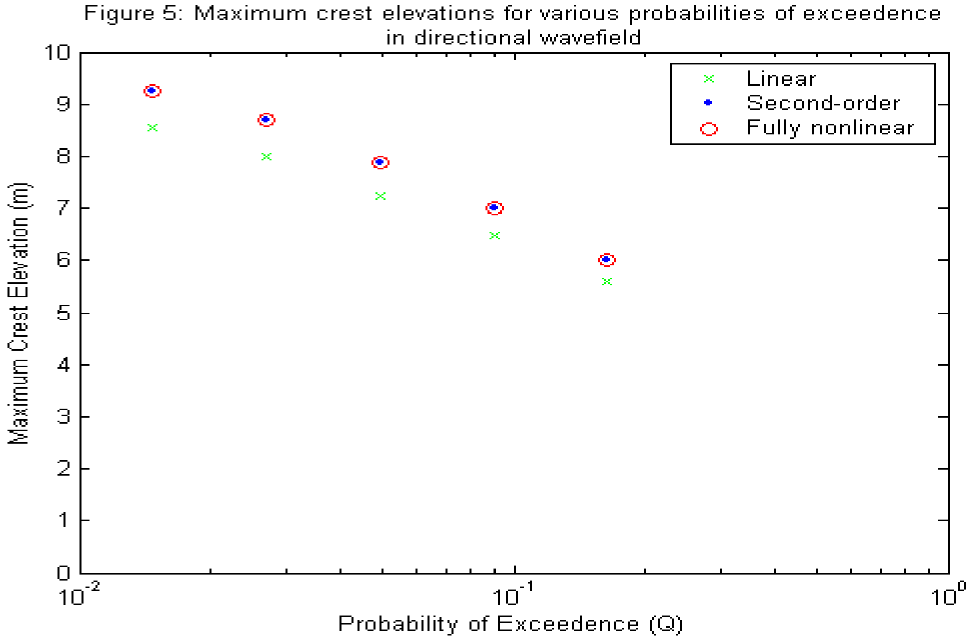


Figure 4b: Comparison between Fourier transforms in space of fully nonlinear surface profiles at time of greatest crest elevation.



We have previously noted in section 3 that laboratory data reported by Johannessen & Swan (2001 & 2002) confirm that these energy transfers are strongly dependent on the underlying directionality. This is further confirmed by the calculations presented on Figure 5. These again describe crest elevations vs exceedence probabilities for the JONSWAP spectrum noted previously, but in this case involve directionally spread waves with a wrapped normal standard deviation of 30º. This shows a marked reduction in nonlinearity with directional spread.



5.0 CLOSING REMARKS

The extent to which the present results are representative of wide-ranging wave conditions remains unclear. It is also important to acknowledge that there remains considerable work to be done in respect of basing the optimisation on the fully nonlinear description of the water surface elevation. Nevertheless, the results are interesting in that they confirm the importance of directionality and suggest that provided a sea state has a significant directional spread, crest elevation statistics based on the second-order theory of Sharma & Dean (1981) may well provide a very realistic representation. However, they also suggest that if the directionality reduces very different conclusions may be drawn. Indeed the results that one possible explanation for the occurrence of unexpectedly large waves (commonly referred to as 'freak' or 'rogue' waves) may be the extreme nonlinear interactions that can arise as the directionality of the wave field reduces.

References

Baldock, T.E., C. Swan and P.H. Taylor, 1996: A laboratory study of non-linear surface waves on water. *Phil Trans. Roy. Soc. London Ser A*, **354**, 649.

Bateman, W.J.D., C. Swan and P.H.Taylor, 2001: On the efficient numerical simulation of directionally spread surface water waves. *Journal of Computational Physics*, **174**, 277-305.

Bateman, W.J.D., C. Swan and P.H.Taylor, 2002: On the calculation of the Water Particle Kinematics arising in a Directionally Spread Wavefield. *In press: Journal of Computational Physics*.

Craig, W. and C. Sulem, 1993: Numerical simulation of gravity waves, *J. Computational Physics*, **108**, 73 (1993).

Fenton, J.D, and Rienecker, M.M. 1982. A Fourier method for solving nonlinear water-wave problems: application to solitary-wave interactions, *J. Fluid Mech*., **118**, 411-443.

Fenton, J.D., 1985: A fifth-order stokes theory for steady waves. *J. of Waterway, Port, Coastal and Ocean Engineering*, **111**(2), 216-234.

Forristall, G.Z., 1998: Wave crest distributions: observations and second order theory. *J. Phys. Oceanography*, **30**, 1931-1943.

Johannessen, T.B. and C.Swan, 2001: A laboratory study of the focusing of transient and directionally spread surface water waves. *Proc. Roy. Soc. Ser. A*. **457**, 971-1006

Johannessen, T.B. and C.Swan, 2002: On the nonlinear dynamics of wave groups produced by the focusing of surface water waves. *In press: Proc. Roy. Soc. Ser. A*.

Jonathan, P. and P.H. Taylor: Irregular nonlinear waves in a spread sea. *ASME Trans. J. Offshore Mech. Arctic Eng*., **119**, 37.

Lingren, G., 1970: Some properties of a Normal process near a local maximum. *The Annals of Mathematical Statistics*, **41**(6), 1870-1883.

Longuet-Higgins, M.S. and E.D. Cokelet, 1976: The deformation of steep surface waves on water. A numerical method of computation. *Proc. R. Soc. Lond., A* **350**, 1-26.

Madsen, H.O., S. Krenk and N.C. Lind, 1986: Methods of structural safety. *Prentice-Hall Inc., Englewood Cliffs, New Jersey, USA*.

Melchers, R.E., 1987: Structural reliability analysis and prediction. *Ellis Horwood Ltd, Chichester , UK*.

Sharma, J.N. and R.G. Dean, 1981: Second-Order Directional Seas and Associated Wave Forces. *Society of Petroleum Engineering Journal*, **4**, pp 129-140.

Tromans, P.S., A. Anaturk and P. Hagemeijer, 1991: A new model for the kinematics of large ocean waves – application as a design wave. *Proc. 1st Offshore and Polar Engineering Conf (ISOPE)*, **3**, 64-71.

Tromans, P.S., J. van Dam, 1996: The probability distribution of extreme load on space frame structures. *Proc. of OMAE Conf., Florence*.

Tromans, P.S. and I.K. Suastika, 1998: Spectral Response Surfaces, Designer Waves and the Ringing of Offshore Structures. *Proc. of the 1998 International OTRC Symposium, Ocean Wave Kinematics, Dynamics and Loads on Structures*, 171-178.

Tromans, P.S. and P.H. Taylor, 1998: The shapes, histories and statistics of non-linear wave crests in random seas. *Proc. of OMAE Conf., Lisbon*.

Tromans, P.S. and L. Vanderschuren, 2002: The statistics of wave crest elevation calculated to second order.