Eliminating Polar Singularities in Global Wave Models

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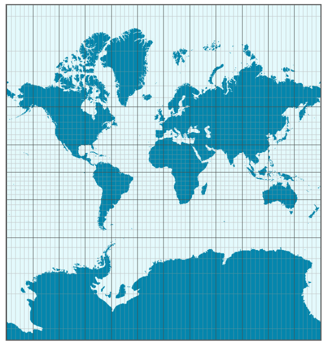
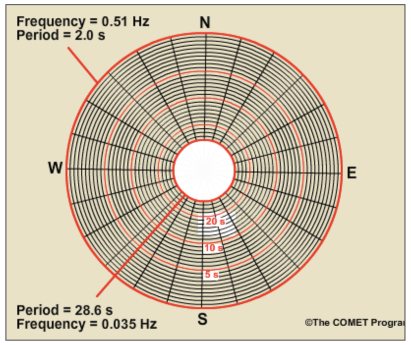
I. Introduction

- Global third-generation spectral wave models use a governing equation in $2\mathrm{D} \times 2\mathrm{D}$ phase space that contains polar singularities in both the spatial and spectral domains.
- Singularities can be circumvented using artificial boundaries or advanced model setups (e.g., structured multi-cell, curvilinear grids).
- However, polar singularities are not inevitable, and a non-singular global wave model can be constructed in $3\mathrm{D} \times 3\mathrm{D}$ phase space.

Spatial



Spectral



- At the poles, spatial coordinates are non-unique and spectral directions are undefined.

II. Polar singularities in 4D phase space

- For simplified kinematic testing, consider an aqua planet without source terms, ocean currents, or bathymetric variations.
- A $2\mathrm{D} \times 2\mathrm{D}$ form of the wave action balance (WAB) equation, using a spatial global Mercator projection and a spectral global tangent plane projection, can be written as:

$$\left\{ \partial_t + \frac{c_g(f)}{R_E} \left(\frac{\cos\theta}{\cos\mu} \partial_\lambda + \sin\theta\, \partial_\mu - \tan\mu\, \cos\theta\, \partial_\theta \right) \right\} \mathscr{W}(\lambda, \mu, f, \theta, t) = 0$$

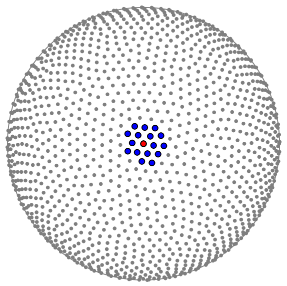
- $\mathscr{W}$: wave action density
- (μ, λ) : latitude and longitude
- R_E : radius of the Earth

V. Numerical method and directional alignment

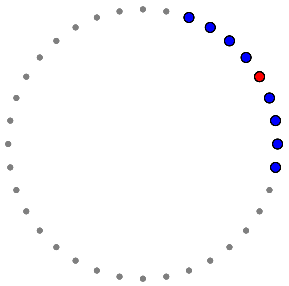
- Approach tested with RBF-FD, a finite difference method on unstructured nodes.
- Directional alignment of the nearest neighbors is defined by their projection onto the local tangent plane for the stencil node.

- PHS+Poly scheme is used to avoid parameter tuning and add stability near boundaries.
- Differential weights are assembled per spatial and directional node using nearest neighbors.
- Wave spectral density for the desired projected direction is interpolated at each neighbor to estimate spatial advection in the same fixed direction.

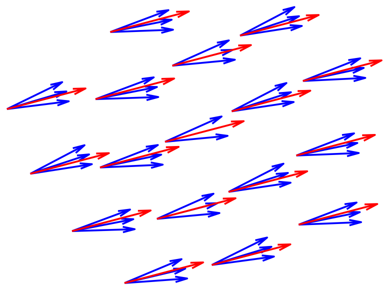
Spatial stencil



Directional stencil

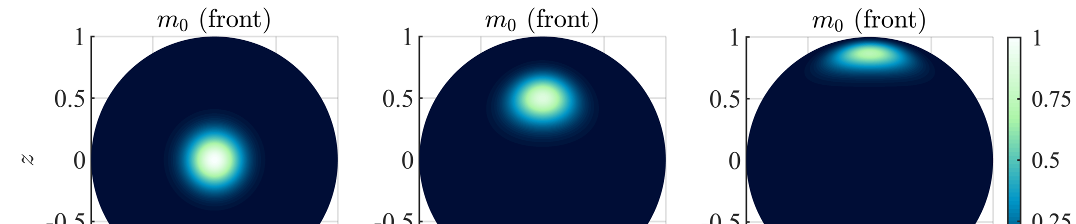


Interpolated direction



VI. Results: Propagation across North Pole

- Initial spatial amplitude: Normalized Gaussian bell (half-width 20°) centered at $(1{,}0{,}0)$.
- Initial spectral distribution: Narrow cosine-power distribution (exponent $=20$, $\approx 60^\circ$ width) with an initial dominant direction of $(0{,}0{,}1)$ at spatial bell center.



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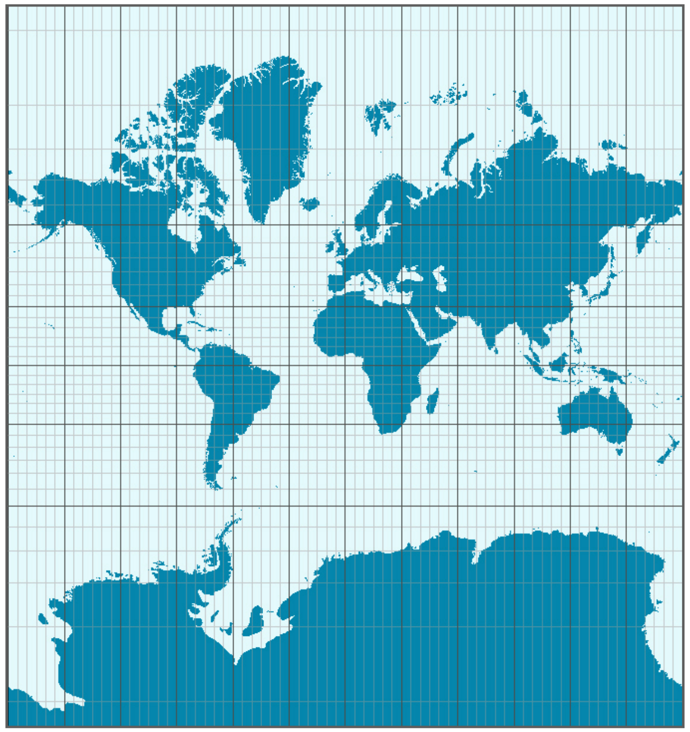

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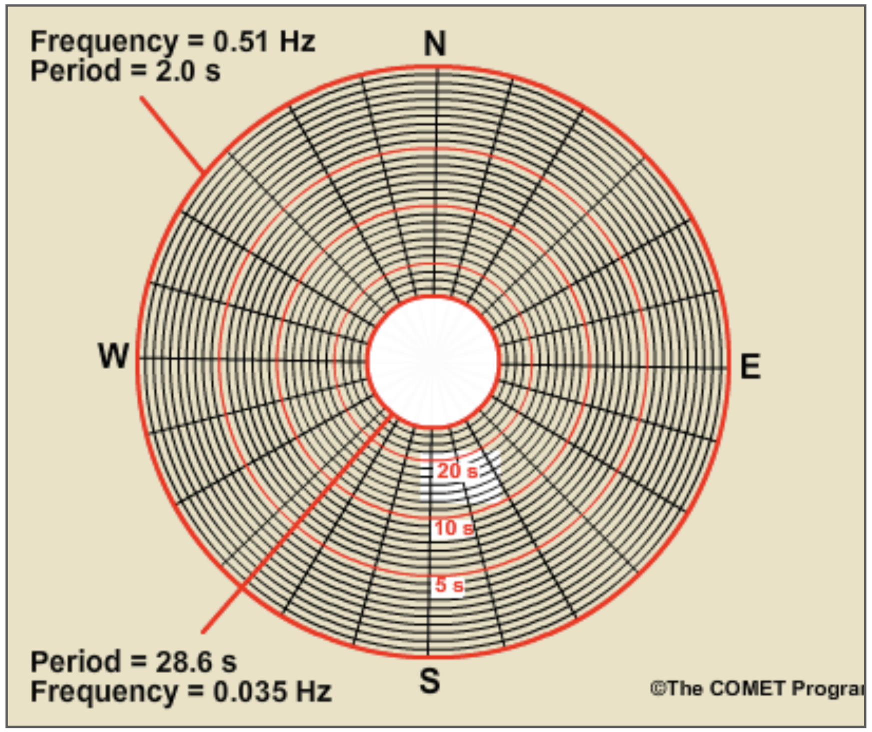

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  - (μ, λ) : latitude and longitude
  - R_E : radius of the Earth
  - c_g : deep-water group velocity
  - (f, θ) : frequency and direction

- Notice that there are transitions across the poles (e.g., at the North Pole, the coordinates shift as $\lambda \Rightarrow \lambda + 180^\circ$, $\mu \Rightarrow 180^\circ - \mu$, and $\theta \Rightarrow -90^\circ$).

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- To avoid introducing singularities from 2D projection mappings, the full 3D spatial and spectral domains are preserved while restricting transport to the relevant 2D manifolds.

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- To avoid introducing singularities from 2D projection mappings, the full 3D spatial and spectral domains are preserved while restricting transport to the relevant 2D manifolds.
- Ensures that each spatial position and spectral direction remains unique without increasing the number of model unknowns.

- Proposed solution (for a general dispersion relation Ω) is of the form

$$\begin{aligned}\partial_t \mathscr{W}(\mathbf{x}, \mathbf{k}, t) \;+\; & P_{\mathbf{k}} \nabla_{\mathbf{k}} \Omega(\mathbf{x}, \mathbf{k}, t) \cdot P_{\mathbf{x}} \nabla_{\mathbf{x}} \mathscr{W}(\mathbf{x}, \mathbf{k}, t) \\ - \; & P_{\mathbf{x}} \nabla_{\mathbf{x}} \Omega(\mathbf{x}, \mathbf{k}, t) \cdot P_{\mathbf{k}} \nabla_{\mathbf{k}} \mathscr{W}(\mathbf{x}, \mathbf{k}, t) = 0\end{aligned}$$

where $P_{\mathbf{x}}$ and $P_{\mathbf{k}}$ act as projection operators for the spatial and spectral domains.

- Following restriction is imposed to ensure the spatial and spectral coordinates are tangent:

$$\mathbf{x} \cdot \mathbf{k} = 0, \quad \text{for } \mathbf{x}, \mathbf{k} \in \mathbb{R}^3$$

IV. Spherical projection operator

- Spherical projection operator is used to **(i)** confine spatial advection to the sphere and **(ii)**

$$\mathbf{x} \cdot \mathbf{k} = 0, \quad \text{for } \mathbf{x}, \mathbf{k} \in \mathbb{R}^3$$

IV. Spherical projection operator

- Spherical projection operator is used to **(i)** confine spatial advection to the sphere and **(ii)** confine spectral evolution to the plane tangent to the spatial location.

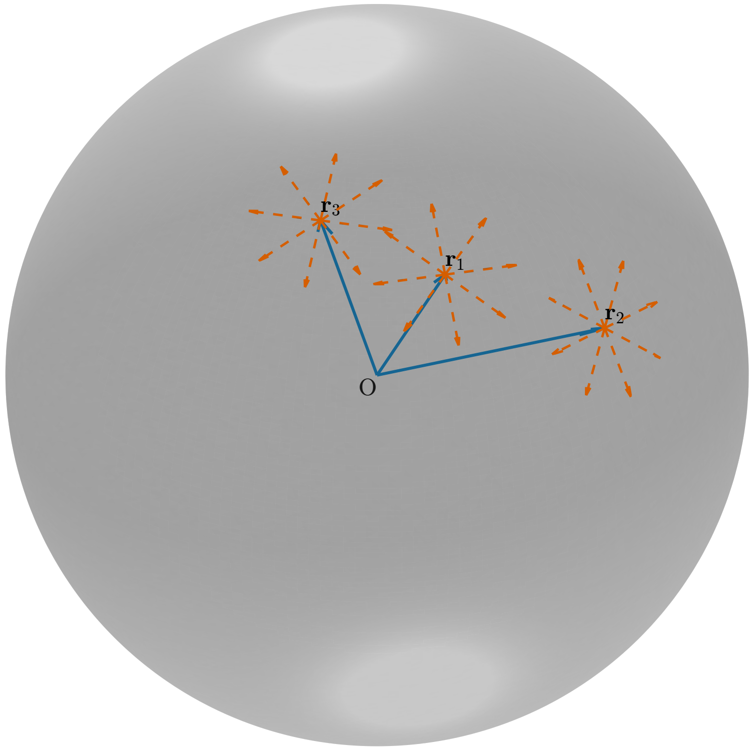
- For any $\boldsymbol{\nu} \in \mathbb{R}^3$, we define $P_{\mathbf{x}}$ (and $P_{\mathbf{k}}$ similarly) as

$$P_{\mathbf{x}} \boldsymbol{\nu} = \boldsymbol{\nu} - \hat{\mathbf{n}} \left(\hat{\mathbf{n}} \cdot \boldsymbol{\nu} \right) = \left(\mathbf{I}_3 - \frac{1}{R_E^2} \mathbf{x}\mathbf{x}^T \right) \boldsymbol{\nu}$$

- In the kinematic test case, the proposed solution simplifies (surprisingly*) to the set of decoupled equations

$$\left\{ \partial_t + \mathbf{c_g}(\mathbf{k}) \cdot \nabla_{\mathbf{x}} \right\} \mathscr{W}(\mathbf{x}, \mathbf{k}, t) = 0$$

for each $\mathbf{k} \in \mathbb{R}^3$ that satisfies $\mathbf{x} \cdot \mathbf{k} = 0$.



Sample local directions

*Projections only fully cancel in the absence of currents or bathymetry variations.

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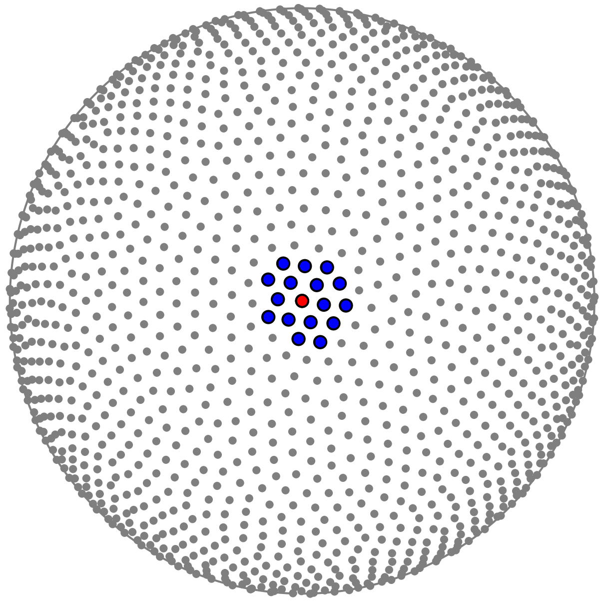
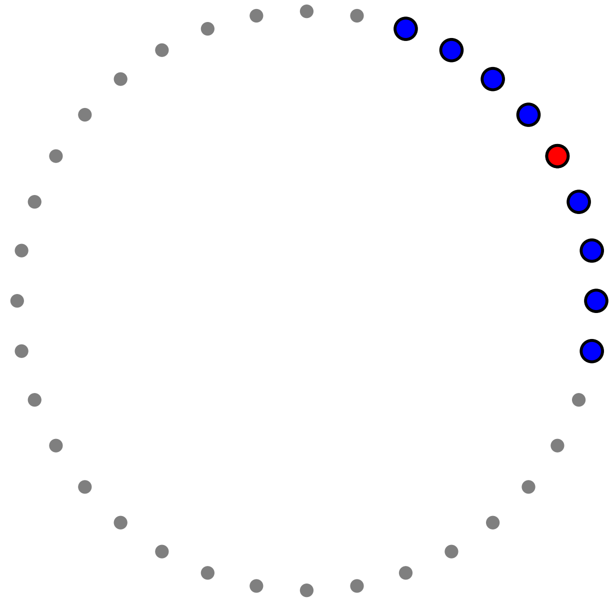
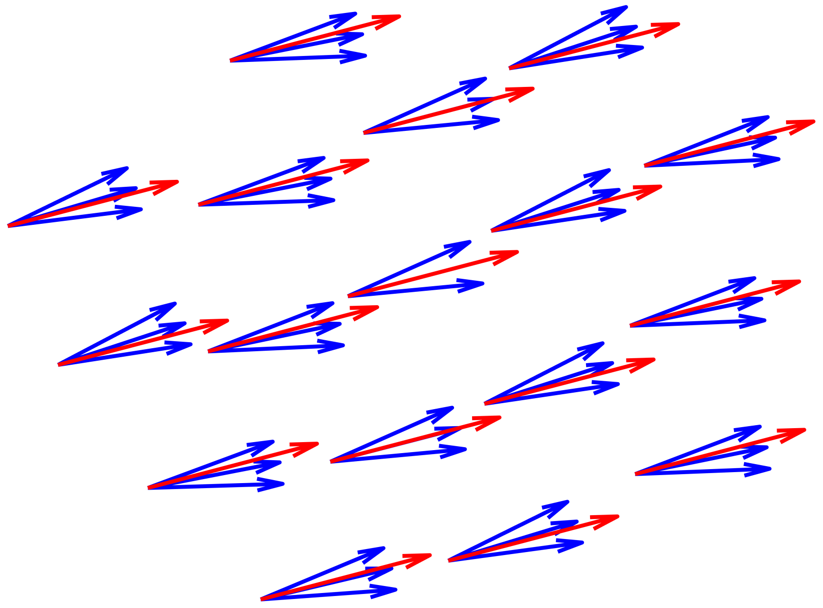


Directional stencil



Interpolated direction



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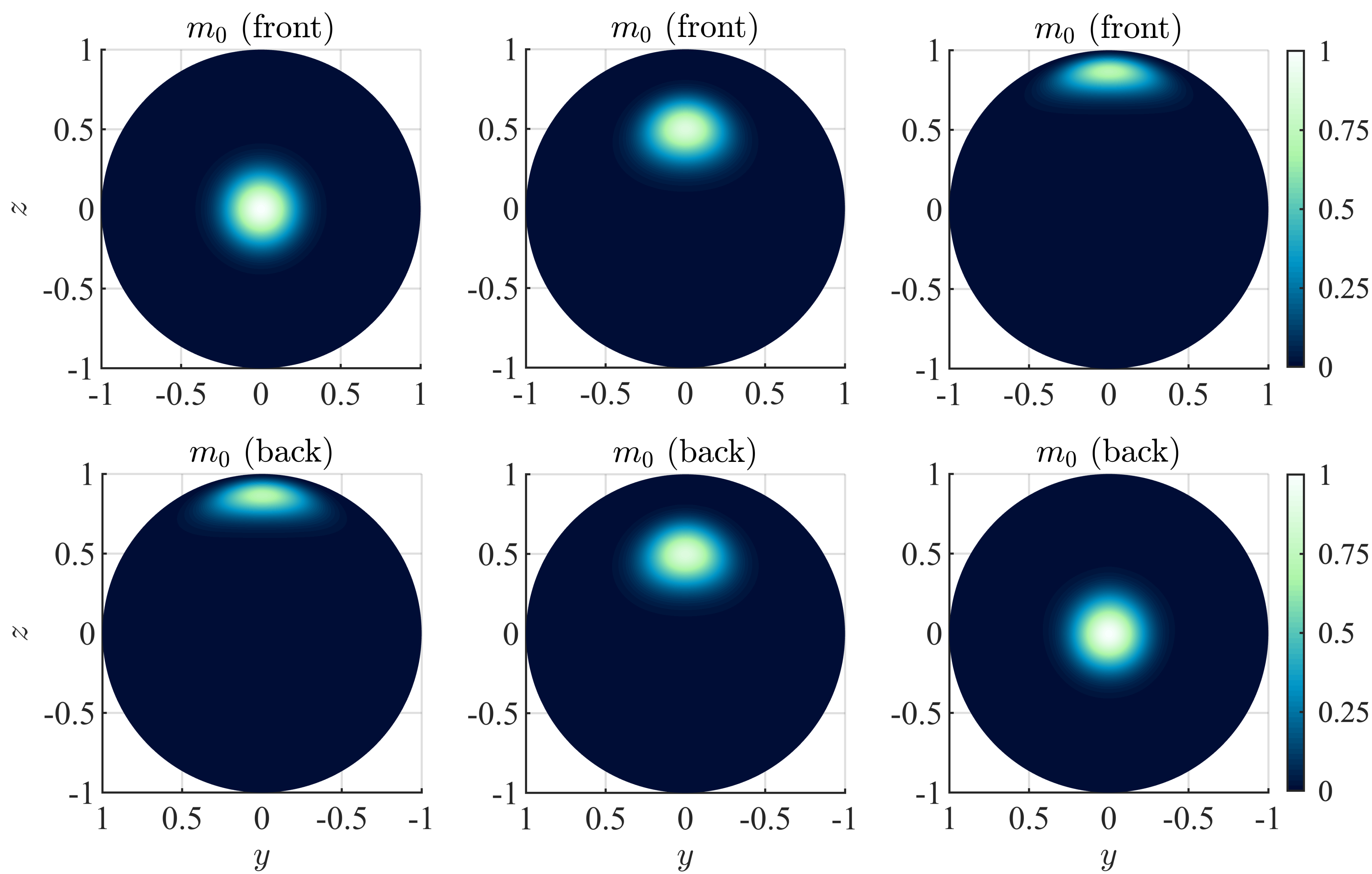
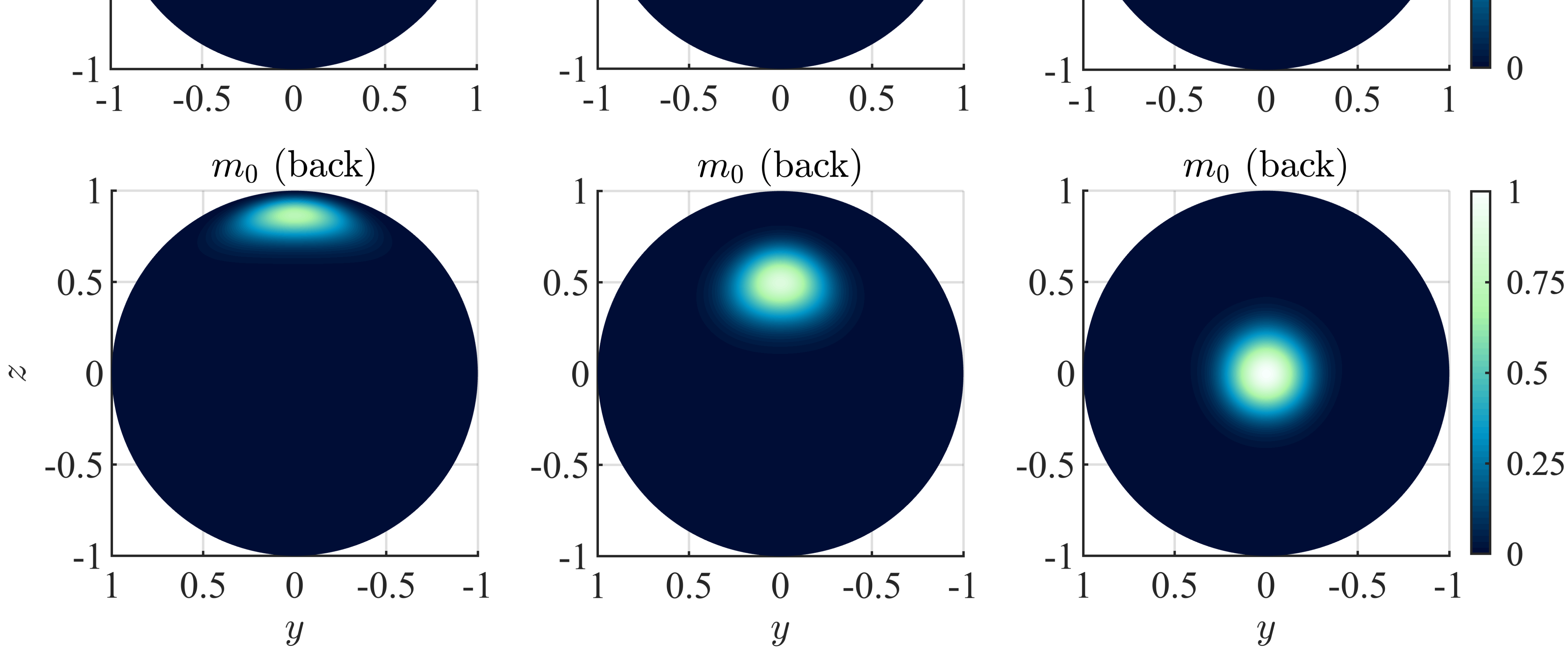
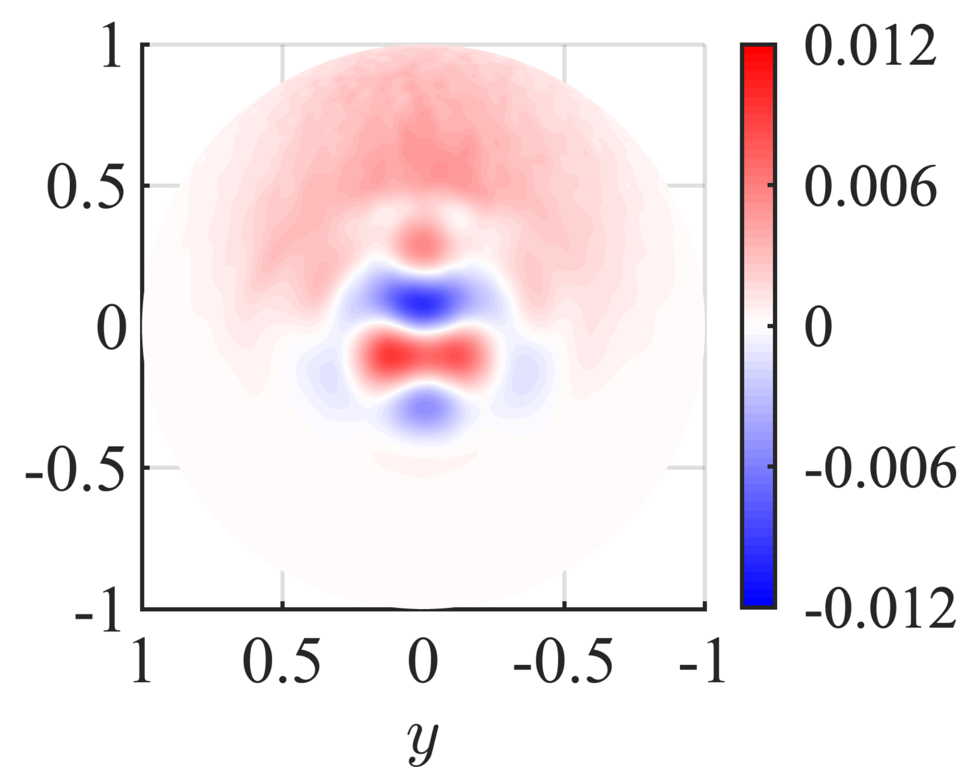




Sample transport of a storm wave system across the North Pole. Zeroth moment (m_0) shown at the following intervals: $(0,1,2,4,5,6)\,T/6$**.**

- Relative residual error after 1/2 revolution shown.
- Model resolution: 4° spatial ($N_{\mathbf{x}} = 8100$, $n_{\mathbf{x}} = 31$), 6° directional ($N_{\theta} = 60$, $n_{\theta} = 9$, $p_{\theta} = 5$).
- Similar errors obtained for different initial directions.
- WAVEWATCH III reference error: $\approx 4\,\%$ for global

  $1^{\circ} \times 1.25^{\circ}$ lat-lon model setup ($N_{\mathbf{x}} = 44000$, $N_{\theta} = 36$).



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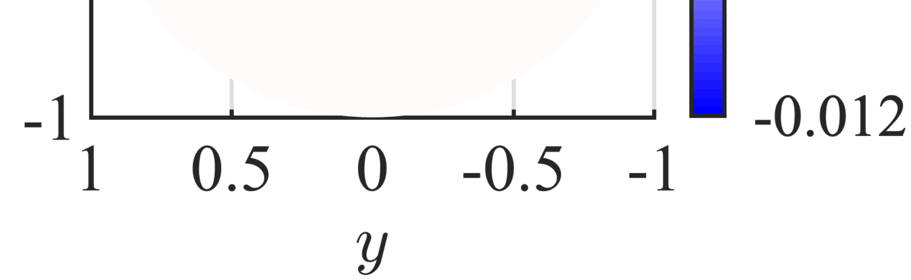

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