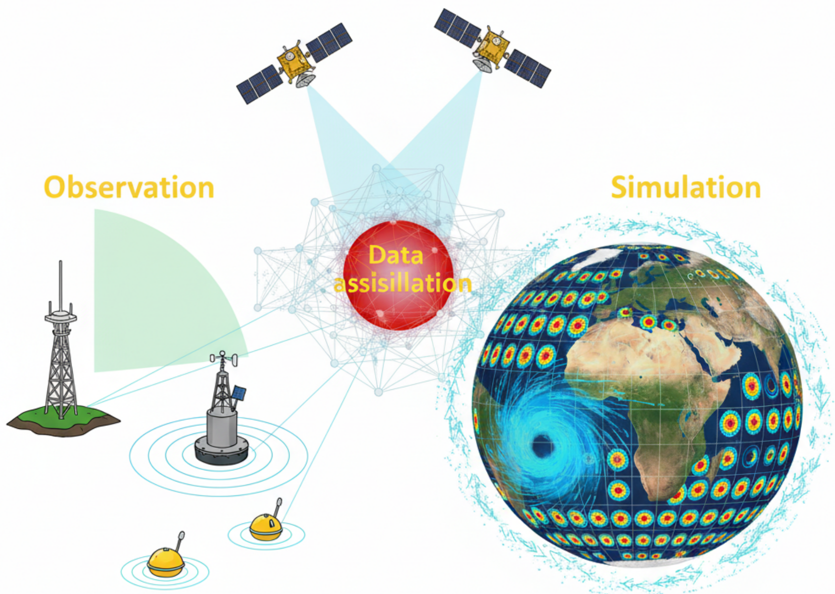
Assimilation of Spectral Wave Buoy Observations by Transformation of the Directional Spectrum Background

Arthur Filoche[1,2], Jeff Hansen[1] and Kevin Vinsen[1,2]

[1] University of Western Australia, Oceans Institute, TIDE
[2] University of Western Australia, ICRAR

Optimally combines observation and Physics simulation

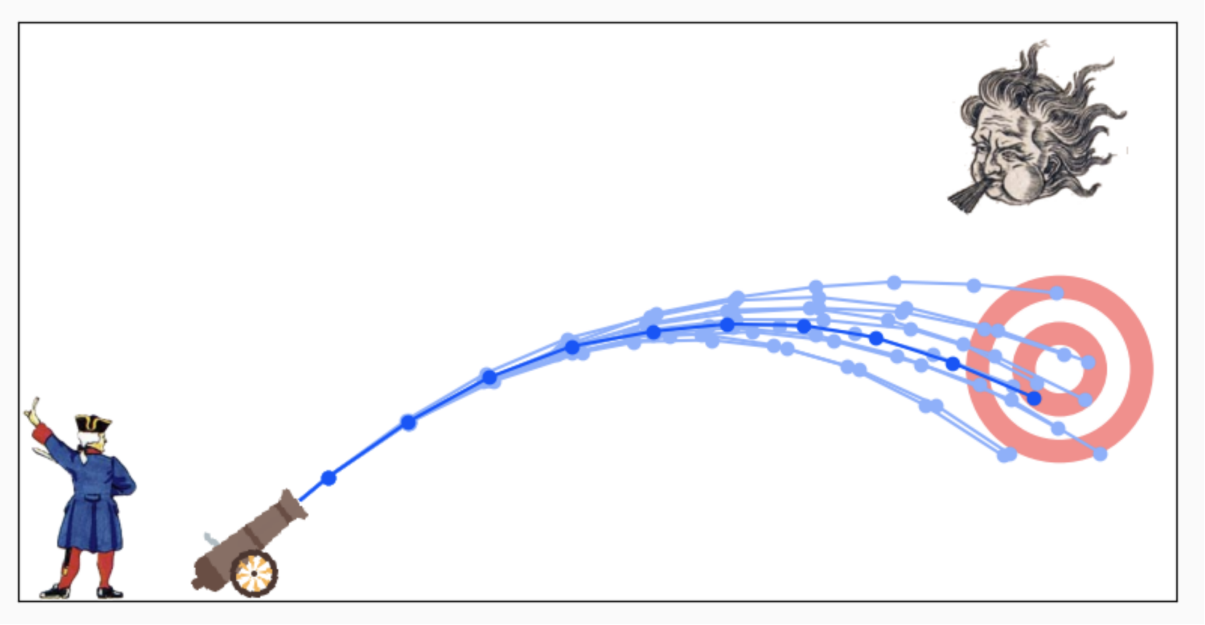


adapted from Ryuji Yoshida illustration (riken.jp), Gemini

Data Assimilation

Optimally combines observation and Physics simulation **to estimate initial condition**

▷ an optimal control problem

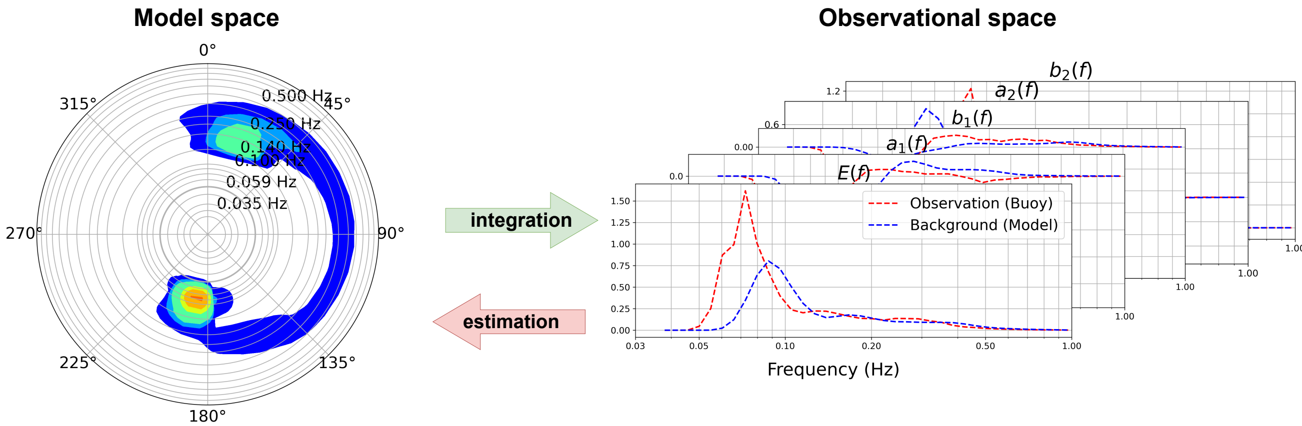


from "Automatic differentiation for applied mathematicians", J.Feydy

Assimilation of spectral wave measurements

Notation:

- ▷ **Model state**: Directional Wave Spectra $\mathbf{X} = E(f, \theta)$
- ▷ **Observation**: first-5 directional Fourier coefficients $\mathbf{Y} = [a_0, a_1, b_1, a_2, b_2](f)$



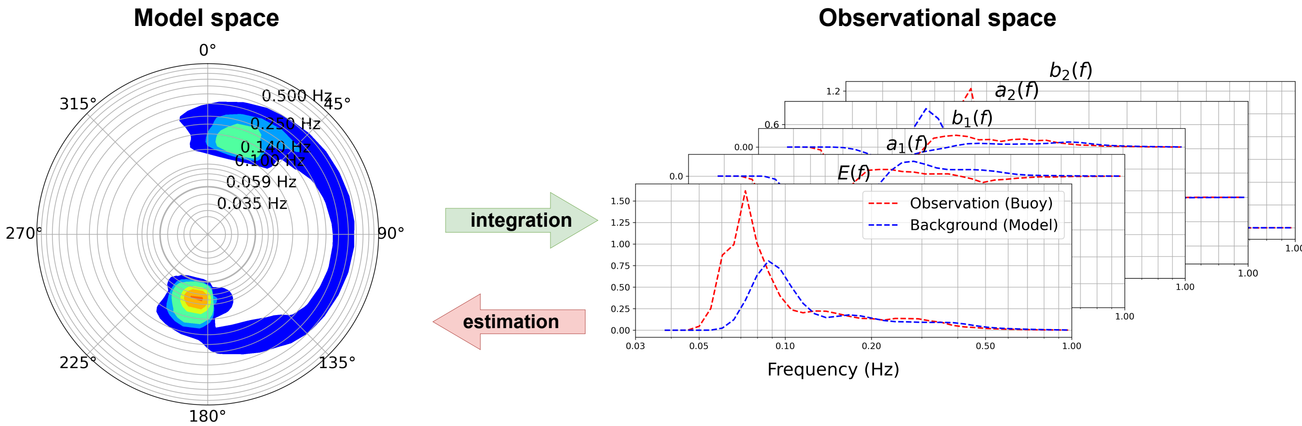
Observation equation: $\mathbf{Y}(f) = \int_0^{2\pi} \mathbf{X}(f, \theta) \odot \begin{bmatrix} 1 & \cos(\theta) & \sin(\theta) & \cos(2\theta) & \sin(2\theta) \end{bmatrix} d\theta$

- ▷ information on a finite number of integral properties

Assimilation of spectral wave measurements

Notation:

- ▷ **Model state**: Directional Wave Spectra $\mathbf{X} = E(f, \theta)$
- ▷ **Observation**: first-5 directional Fourier coefficients $\mathbf{Y} = [a_0, a_1, b_1, a_2, b_2](f)$



Observation equation: $\mathbf{Y}(f) = \int_0^{2\pi} \mathbf{X}(f, \theta) \odot \left[1 \quad \cos(\theta) \quad \sin(\theta) \quad \cos(2\theta) \quad \sin(2\theta)\right] d\theta$

- ▷ information on a finite number of integral properties

Goal

Correct the model state using information from observation

Succinct literature review

Assimilation of bulk parameters

[Lionello et al., 1992] [Janssen et al., 2008], [Lefevre et al., 2012], [Smit et al., 2021]

- ▷ step 1: assimilation in bulk parameters (observation) space
- ▷ step 2: **geometrical update of the model** $E(f, \theta) \rightarrow \alpha E(\beta f, \theta + \gamma)$

Succinct literature review

Assimilation of bulk parameters

[Lionello et al., 1992] [Janssen et al., 2008], [Lefevre et al., 2012], [Smit et al., 2021]

- ▷ step 1: assimilation in bulk parameters (observation) space
- ▷ step 2: **geometrical update of the model** $E(f, \theta) \rightarrow \alpha E(\beta f, \theta + \gamma)$

Partition-based assimilation of bulk parameters

[Voorips et al., 1992, 1997] [Aouf et al. 2012, 2021]

- ▷ step 1: Spectral partitioning of model and observation
- ▷ step 2: Cross-assignment of wave partitions
- ▷ step 3: Assimilation of wave partitions
- ▷ step 4: partition-wise **geometrical update of the model**

Succinct literature review

Assimilation of bulk parameters

[Lionello et al., 1992] [Janssen et al., 2008], [Lefevre et al., 2012], [Smit et al., 2021]

- ▷ step 1: assimilation in bulk parameters (observation) space
- ▷ step 2: **geometrical update of the model** $E(f, \theta) \to \alpha E(\beta f, \theta + \gamma)$

Partition-based assimilation of bulk parameters

[Voorips et al., 1992, 1997] [Aouf et al. 2012, 2021]

- ▷ step 1: Spectral partitioning of model and observation
- ▷ step 2: Cross-assignment of wave partitions
- ▷ step 3: Assimilation of wave partitions
- ▷ step 4: partition-wise **geometrical update of the model**

Assimilation of first-5 directional Fourier coefficients

[Houghton et al., 2022]

- ▷ step 1: assimilation in directional Fourier coefficients (observation) space
- ▷ step 2: optimization-based **update of the model**

Succinct literature review

Assimilation of bulk parameters

[Lionello et al., 1992] [Janssen et al., 2008], [Lefevre et al., 2012], [Smit et al., 2021]

- ▷ step 1: assimilation in bulk parameters (observation) space
- ▷ step 2: **geometrical update of the model** $E(f, \theta) \rightarrow \alpha E(\beta f, \theta + \gamma)$

Partition-based assimilation of bulk parameters

[Voorips et al., 1992, 1997] [Aouf et al. 2012, 2021]

- ▷ step 1: Spectral partitioning of model and observation
- ▷ step 2: Cross-assignment of wave partitions
- ▷ step 3: Assimilation of wave partitions
- ▷ step 4: partition-wise **geometrical update of the model**

Assimilation of first-5 directional Fourier coefficients

[Houghton et al., 2022]

- ▷ step 1: assimilation in directional Fourier coefficients (observation) space
- ▷ step 2: optimization-based **update of the model**

Motivation

Can we Assimilate in one step and preserve model geometry ?

Classical framework - additive Gaussian errors of **known covariance**

- ▷ Observations: $\mathbf{Y} = \mathbb{H}(\mathbf{X}) + \varepsilon_R \quad$ s.t. $\varepsilon_R \sim \mathcal{N}(0, \mathbf{R})$
- ▷ Background: $\mathbf{X} = \mathbf{X}_B + \varepsilon_B \quad$ s.t. $\varepsilon_R \sim \mathcal{N}(0, \mathbf{B})$

Data Assimilation Framework

Classical framework - additive Gaussian errors of **known covariance**

- ▷ Observations: $\mathbf{Y} = \mathbb{H}(\mathbf{X}) + \varepsilon_R \quad \text{s.t. } \varepsilon_R \sim \mathcal{N}(0, \mathbf{R})$
- ▷ Background: $\mathbf{X} = \mathbf{X}_B + \varepsilon_B \quad \text{s.t. } \varepsilon_R \sim \mathcal{N}(0, \mathbf{B})$

Bayesian Inversion - Maximum A Posteriori (MAP) estimation

- ▷ Variational formulation

$$\arg\max_{\mathbf{X}} p(\mathbf{X} \mid \mathbf{Y}) = \arg\min_{\mathbf{X}} \quad \frac{1}{2} \underbrace{\|\mathbf{X} - \mathbf{X}_B\|^2_{\mathbf{B}}}_{\text{fit-to-prior}} + \frac{1}{2} \underbrace{\|\mathbb{H}(\mathbf{X}) - \mathbf{Y}\|^2_{\mathbf{R}}}_{\text{fit-to-data}}$$

Classical framework - additive Gaussian errors of **known covariance**

- ▷ Observations: $\mathbf{Y} = \mathbb{H}(\mathbf{X}) + \varepsilon_R \quad \text{s.t. } \varepsilon_R \sim \mathcal{N}(0, \mathbf{R})$
- ▷ Background: $\mathbf{X} = \mathbf{X}_B + \varepsilon_B \quad \text{s.t. } \varepsilon_R \sim \mathcal{N}(0, \mathbf{B})$

Bayesian Inversion - Maximum A Posteriori (MAP) estimation

- ▷ Variational formulation

$$\arg\max_{\mathbf{X}} p(\mathbf{X} \mid \mathbf{Y}) = \arg\min_{\mathbf{X}} \quad \frac{1}{2}\underbrace{\|\mathbf{X} - \mathbf{X}_B\|^2_{\mathbf{B}}}_{\text{fit-to-prior}} + \frac{1}{2}\underbrace{\|\mathbb{H}(\mathbf{X}) - \mathbf{Y}\|^2_{\mathbf{R}}}_{\text{fit-to-data}}$$

Why B may be problematic ?

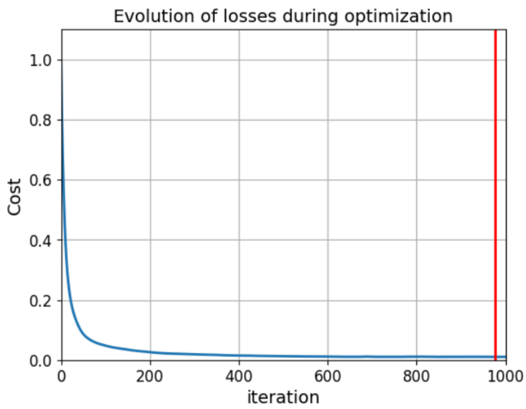
- ▷ High dimensional: $\dim(\mathbf{B}) = (n_f \times n_\theta \times n_x \times n_y)^2$
- ▷ Hard to estimate: ε_B is never measured
- ▷ 2-step processes simplify **B** specification (only spatial covariance)

B is critical for Variational Inversion

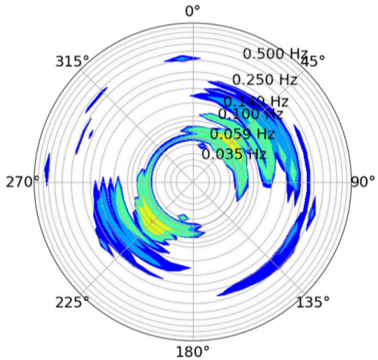
[Long et al., 1979][Ochoa et al., 1990][Crosby et al., 2017]

Example of Assimilation without background regularization

observational loss



estimated spectra



▷ The model background only serves as starting point of the optimization

▷ Observational loss is well optimized but the estimation is distorted

Assimilation as transformation of the model background

Proposed framework

- ▷ Observations: $\mathbf{Y} = \mathbb{H}(\mathbf{X}) + \varepsilon_R \quad \text{s.t. } \varepsilon_R \sim \mathcal{N}(0, \mathbf{R})$
- ▷ Background transformation: $\mathbf{X} = \mathcal{T}_\phi(\mathbf{X}_B)$

Assimilation as transformation of the model background

Proposed framework

- ▷ Observations: $\mathbf{Y} = \mathbb{H}(\mathbf{X}) + \varepsilon_R \quad \text{s.t. } \varepsilon_R \sim \mathcal{N}(0, \mathbf{R})$
- ▷ Background transformation: $\mathbf{X} = \mathcal{T}_\phi(\mathbf{X}_B)$

Bayesian Inversion - Maximum Likelihood Estimation (MLE)

- ▷ Variational formulation

$$\arg\max_{\mathbf{X}} p(\mathbf{X} \mid \mathbf{Y}) = \arg\min_{\phi} \quad \frac{1}{2} \underbrace{\|\mathbb{H}(\mathbf{X}) - \mathbf{Y}\|^2_{\mathbf{R}}}_{\text{fit-to-data}} \quad \text{s.t.} \quad \mathbf{X} = \mathcal{T}_\phi(\mathbf{X}_B)$$

- ▷ MAP extension: regularization of transformation parameters

Assimilation as transformation of the model background

Proposed framework

- ▷ Observations: $\mathbf{Y} = \mathbb{H}(\mathbf{X}) + \varepsilon_R \quad \text{s.t. } \varepsilon_R \sim \mathcal{N}(0, \mathbf{R})$
- ▷ Background transformation: $\mathbf{X} = \mathcal{T}_\phi(\mathbf{X}_B)$

Bayesian Inversion - Maximum Likelihood Estimation (MLE)

- ▷ Variational formulation

$$\arg\max_{\mathbf{X}} p(\mathbf{X} \mid \mathbf{Y}) = \arg\min_{\phi} \quad \frac{1}{2} \underbrace{\|\mathbb{H}(\mathbf{X}) - \mathbf{Y}\|^2_{\mathbf{R}}}_{\text{fit-to-data}} \quad \text{s.t.} \quad \mathbf{X} = \mathcal{T}_\phi(\mathbf{X}_B)$$

- ▷ MAP extension: regularization of transformation parameters

Exactly as JONSWAP fitting

- ▷ the shape is given by the model input
- ▷ the parameters are the one from the transformation
- ▷ **motivation**: model shape and chosen transformation can regularize the estimation

Assimilation as transformation of the model background

Proposed framework

- ▷ Observations: $\mathbf{Y} = \mathbb{H}(\mathbf{X}) + \varepsilon_R \quad \text{s.t. } \varepsilon_R \sim \mathcal{N}(0, \mathbf{R})$
- ▷ Background transformation: $\mathbf{X} = \mathcal{T}_\phi(\mathbf{X}_B)$

Bayesian Inversion - Maximum Likelihood Estimation (MLE)

- ▷ Variational formulation

$$\arg\max_{\mathbf{X}} p(\mathbf{X} \mid \mathbf{Y}) = \arg\min_{\phi} \quad \frac{1}{2} \underbrace{\|\mathbb{H}(\mathbf{X}) - \mathbf{Y}\|^2_{\mathbf{R}}}_{\text{fit-to-data}} \quad \text{s.t.} \quad \mathbf{X} = \mathcal{T}_\phi(\mathbf{X}_B)$$

- ▷ MAP extension: regularization of transformation parameters

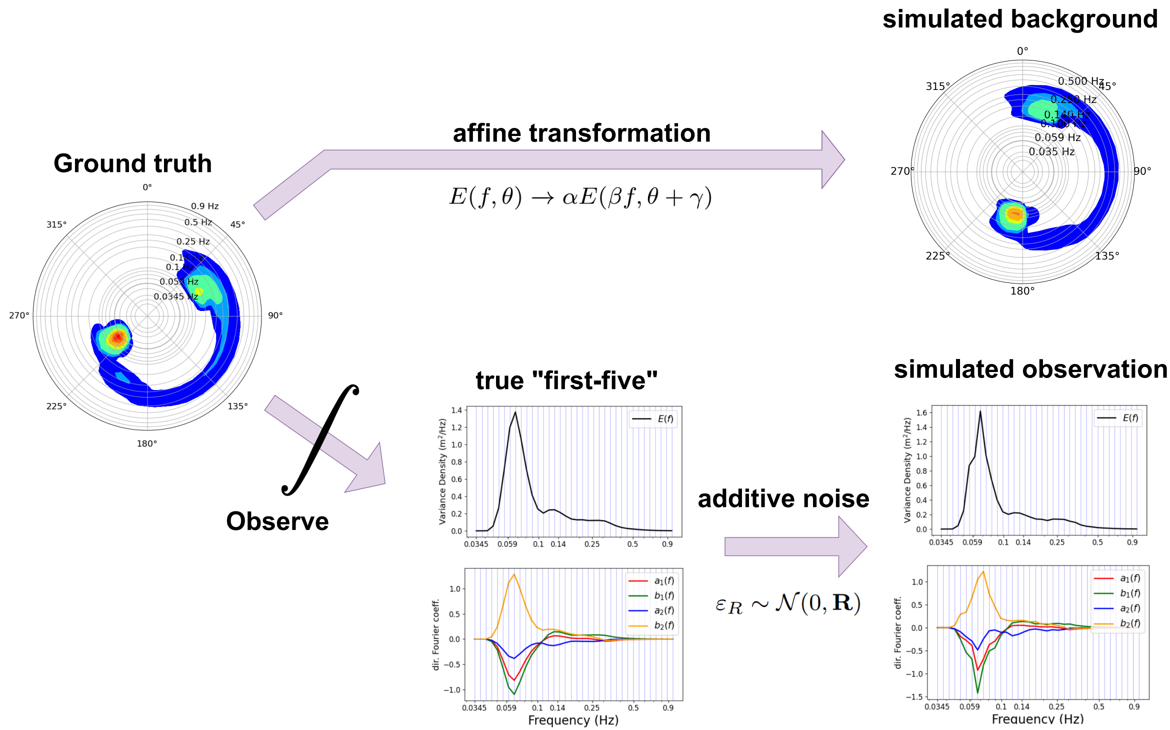
Exactly as JONSWAP fitting

- ▷ the shape is given by the model input
- ▷ the parameters are the one from the transformation
- ▷ **motivation**: model shape and chosen transformation can regularize the estimation

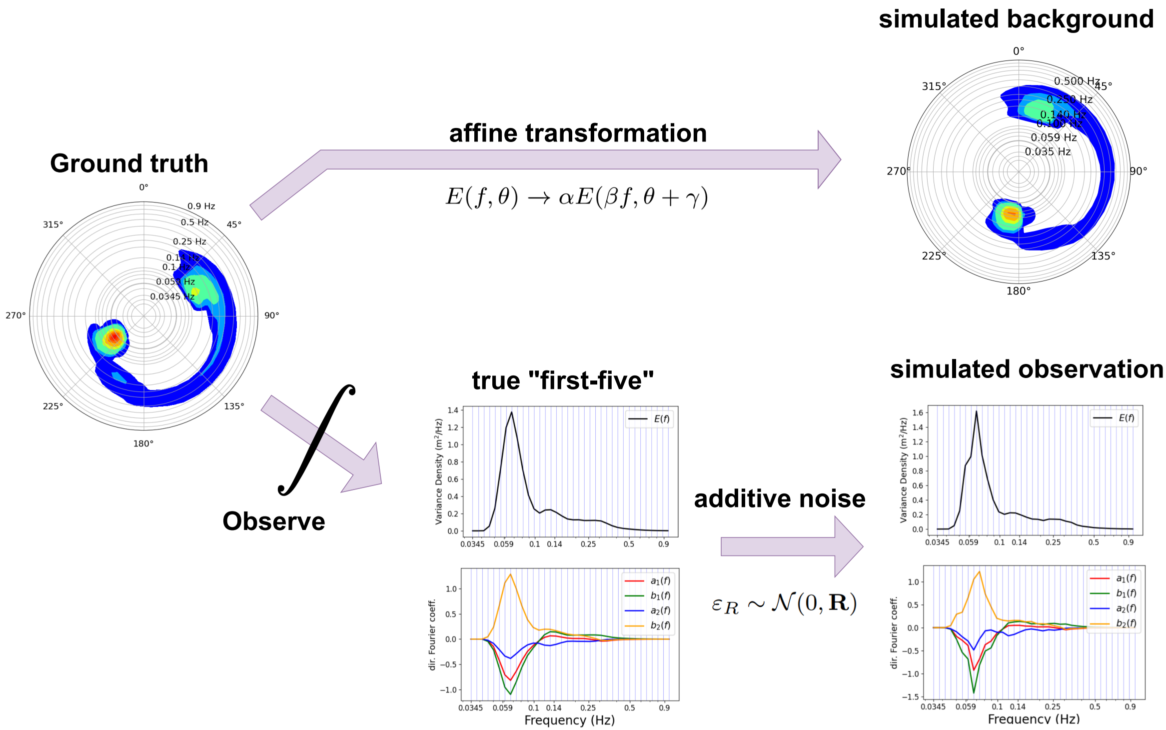
Affine transformation example: $\mathbf{X}(f, \theta) = \phi_1 \mathbf{X}_B(\phi_2 f, \theta + \phi_3)$

- ▷ bilinear interpolation back on the grid after transformation

Synthetic Experiment



Synthetic Experiment

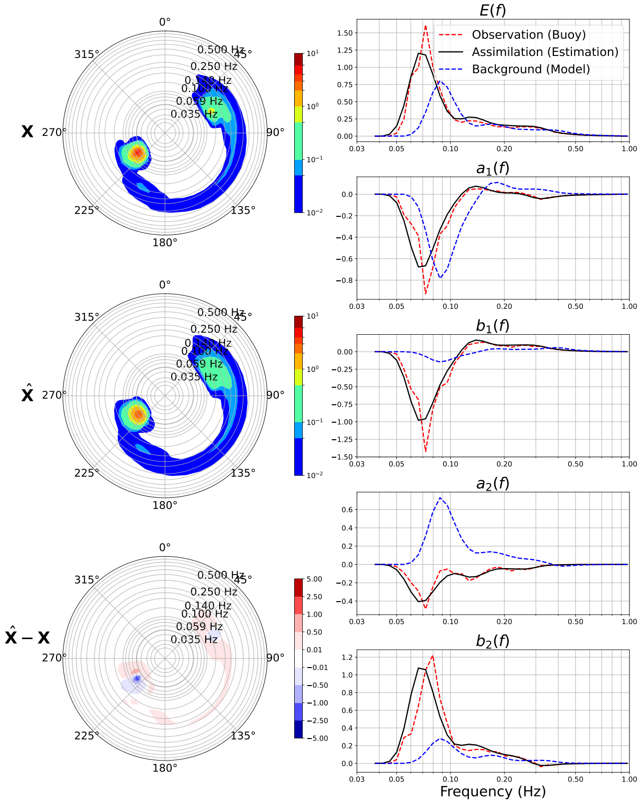


Sanity check

Can I recover the transformation parameters using MLE?

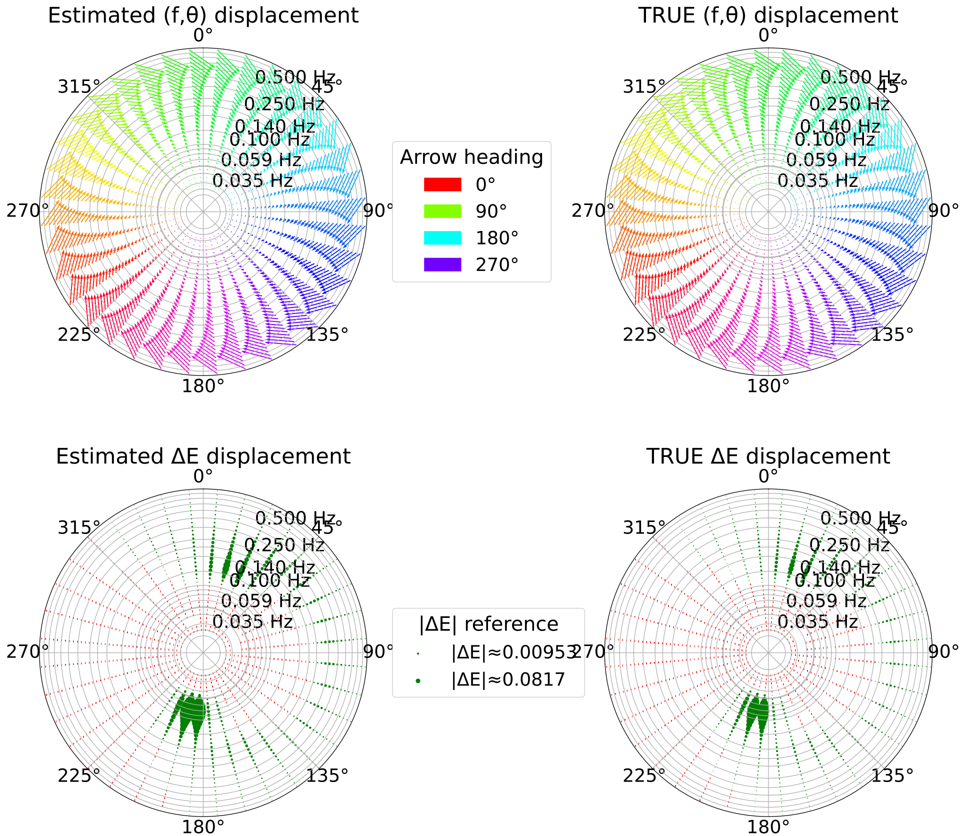
Optimization results

- ▷ Directional Wave Spectrum is transformed to fit the measurements
- ▷ shape+transformation constraints can artificially bump energy



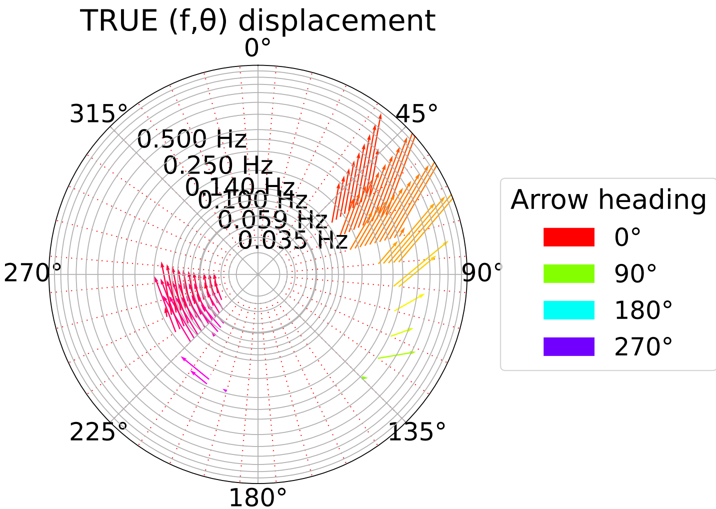
Optimization results

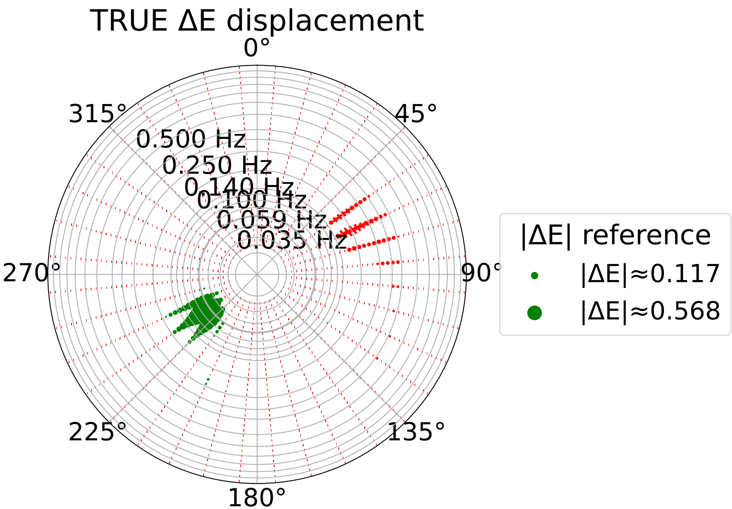
▷ 3D-displacements of the recovered transformation



A more interesting synthetic example

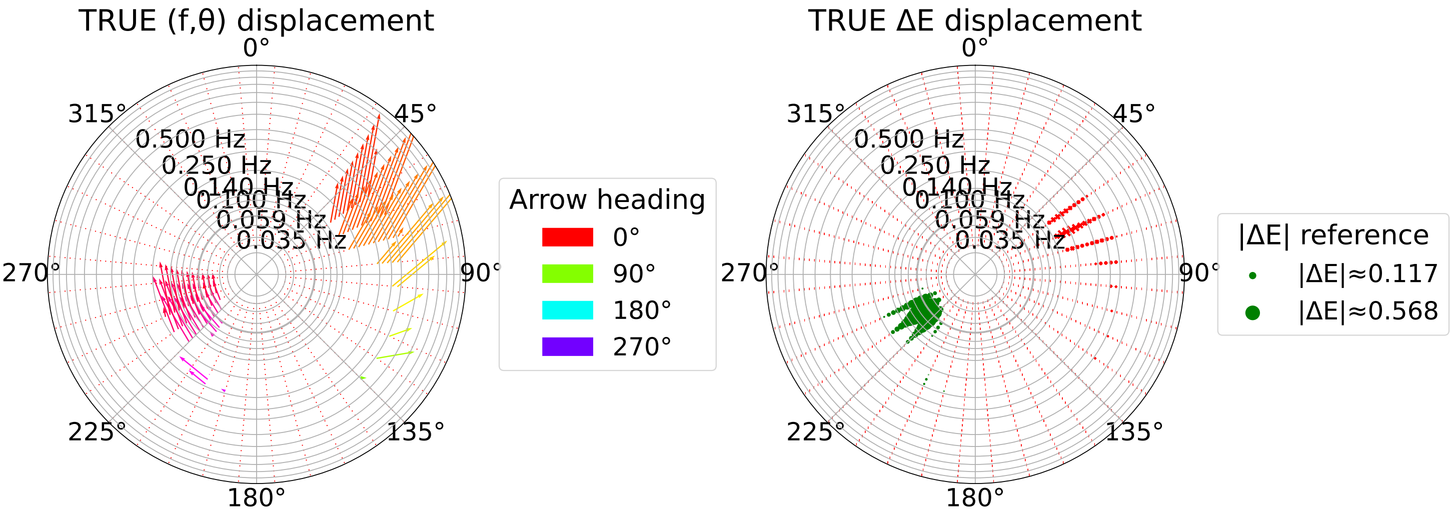
- ▷ **locally affine** transformation for each partition
- ▷ Rotation, scaling, shifting in **opposite direction**





A more interesting synthetic example

- **locally affine** transformation for each partition
- Rotation, scaling, shifting in **opposite direction**



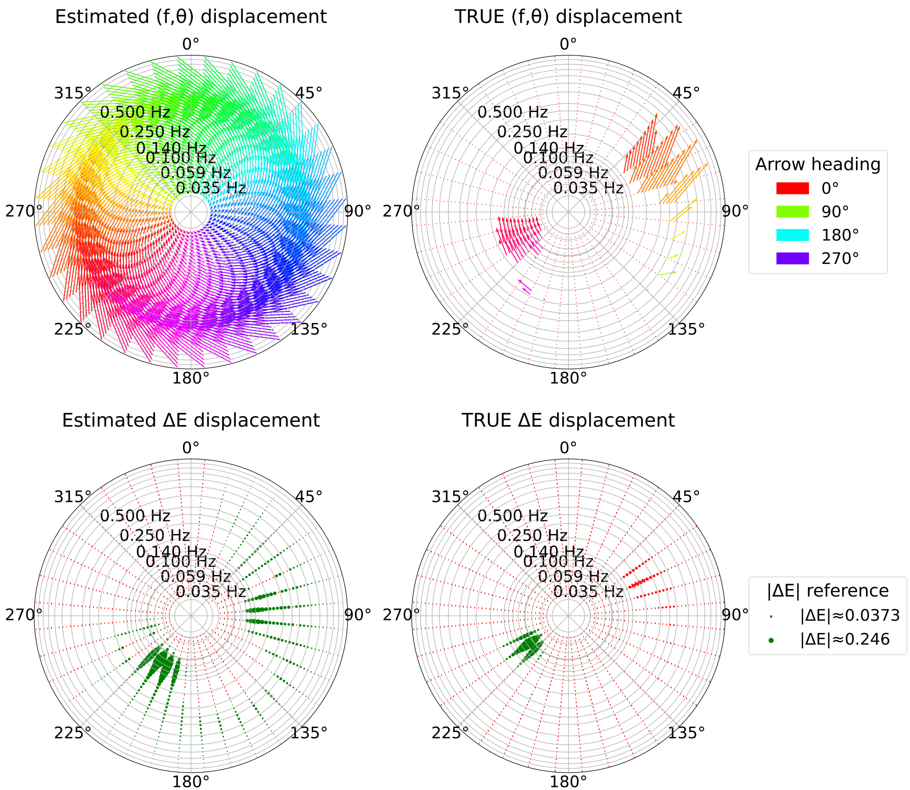
Ideally

Automate partition-wise transformation

Fitting an affine transformation

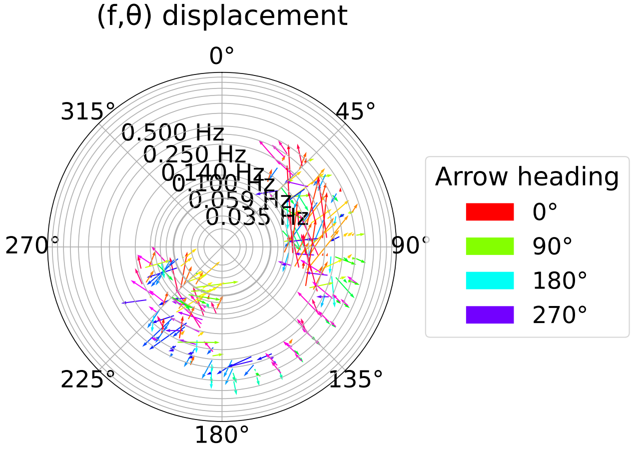
▷ no hope to solve this problem

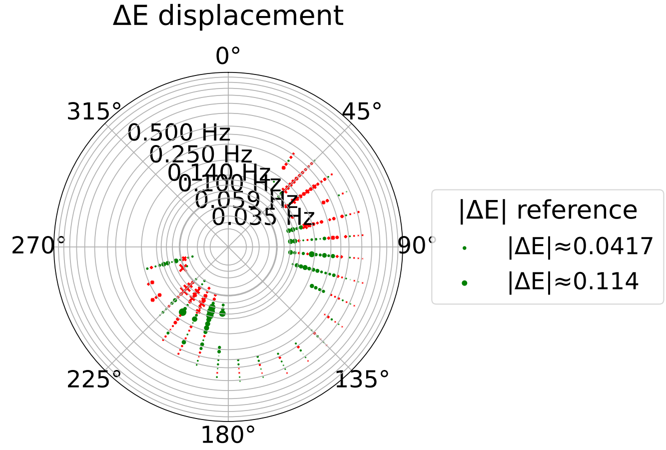
▷ Energy-intense partition impose rotation, scaling and shifting



3D Dense Displacements

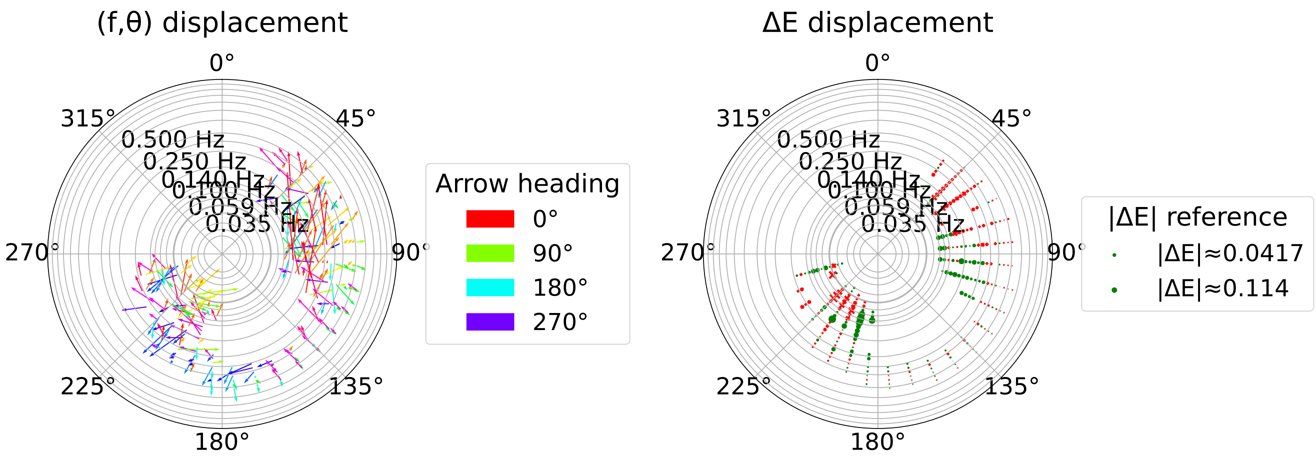
- ▷ **Vector field**: one vector per bin
- ▷ **Optical-flow-like**: only between 2 frames and the second one is not fully observed
- ▷ **ill-posed**: Many displacement fields can explain the data





3D Dense Displacements

- **Vector field**: one vector per bin
- **Optical-flow-like**: only between 2 frames and the second one is not fully observed
- **ill-posed**: Many displacement fields can explain the data

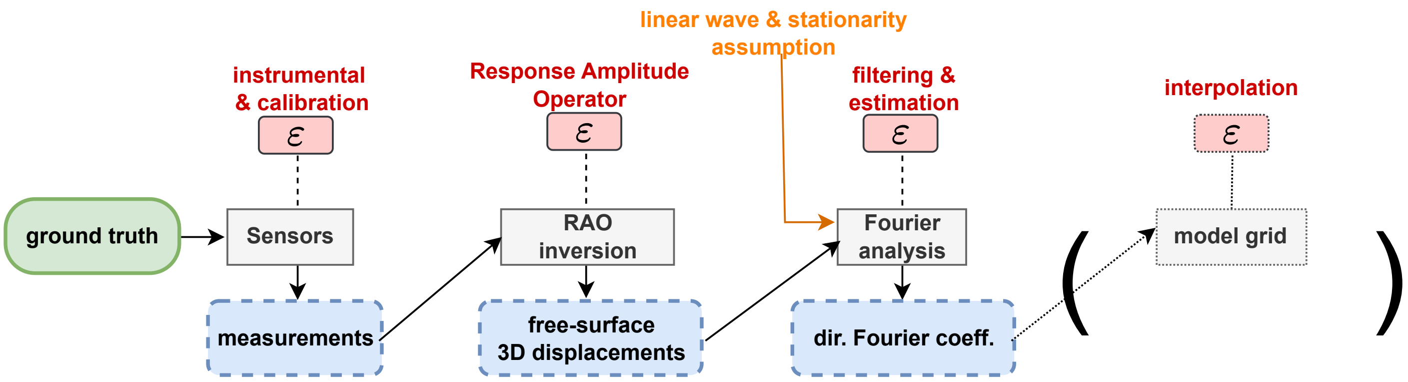


Ironic situation

- state dimension is multiplied by 3
- back to handcraft regularization

What about the observation error covariance R ?

- complex series of operations from raw sea state measurements to estimated coefficients
- used **R** only factors in the spectral density estimation uncertainty



Take home message

You can **update** a model Directional Wave Sectra

directly from under-sampled measurements

using MLE/MAP over the parameters of a **transformation**

arthur.filoche@uwa.edu.au

Observation error covariance $\mathbf{R}$

[Long et al.(1980)][Borgman et al.(1982)]

Observation error covariance $\mathbf{R}$

- Gaussian error model for observation vector $\mathbf{Y}(f) = [a_0,\, a_1,\, b_1,\, a_2,\, b_2]^\top$
- $\mathbf{R} = \mathrm{blockdiag}(R(f_1), \ldots, R(f_N))$, one $5{\times}5$ block per frequency bin
- Effective degree of freedom.: $n_d(f) = \frac{c_{\mathrm{win}} M}{2}\, k_{\mathrm{bins}}(f), \quad k_{\mathrm{bins}}(f) \propto \frac{\Delta f}{1/T}$
- All entries $\propto 1/n_d(f)$; off–diagonals from Q_{zx}, Q_{zy}, C_{xy}

Summary block (symmetric)

$$R(f) = \frac{1}{n_d(f)} \begin{pmatrix} a_0^2 & 0 & 0 & 0 & 0 \\ & \frac{1}{2}(a_0 S_{xx} + Q_{zx}^2) & \frac{1}{2}(Q_{zx}Q_{zy} + a_0 C_{xy}) & \frac{1}{2}Q_{zx}(S_{xx} - S_{yy}) & \frac{1}{2}(Q_{zx}C_{xy} + a_0 Q_{zy}) \\ & \cdot & \frac{1}{2}(a_0 S_{yy} + Q_{zy}^2) & \frac{1}{2}Q_{zy}(S_{yy} - S_{xx}) & \frac{1}{2}(Q_{zy}C_{xy} + a_0 Q_{zx}) \\ & \cdot & \cdot & S_{xx}^2 + S_{yy}^2 - \gamma^2 S_{xx}S_{yy} & (S_{yy} - S_{xx})C_{xy} \\ & \cdot & \cdot & \cdot & 2(S_{xx}S_{yy} + C_{xy}^2) \end{pmatrix}$$

$$\gamma^2 = \frac{C_{xy}^2}{S_{xx}S_{yy}} \leq 1$$

Key points

- Block–diagonal across $f \Rightarrow$ bins independent (“white” across f)
- Within-bin components of $\mathbf{Y}(f)$ correlated (non-diagonal $5{\times}5$)
- Wider $\Delta f \Rightarrow$ larger $n_d(f) \Rightarrow$ smaller variances/covariances
- $\mathbf{R}$ enforced SPD via eigenvalue clamping + Cholesky retry

Possible spatial extension

- Φ associates **each grid point** (x, y) with an **affine transformation** $\Phi_{x,y}$.

$$\min_{\Phi} \mathcal{J}(\Phi) = \underbrace{\sum_{obs} \|\mathbb{H} \circ \mathcal{T}_{\Phi_{obs}}(\mathbf{X}_b) - \mathbf{Y}\|^2_{\mathbf{R}}}_{\text{fit-to-observation}} + \underbrace{\sum_{i,j} \|\Phi_{i,j} - \mathbf{I}_d\|^2_{\mathbf{B}}}_{\text{fit-to-background}}$$

- **B propagates in space the perturbation** from the background $\Phi_b = \mathbf{I}_d$.
- Control space is the transfromation parameter space

Common spatial covariance:

$\triangleright$ $\mathbf{B}_{ij} \propto e^{\left(\frac{-d_{ij}}{\lambda}\right)^p}$