4TH INTERNATIONAL WORKSHOP ON WAVES, STORM SURGES, AND COASTAL HAZARDS
Incorporating the 18th International Waves Workshop

DPRI-KU

PARI

Non-linear Waves in Two-Dimensional Directional Crossing Seas with Bottom Topography Change

Zuorui Lyu[1], Nobuhito Mori[1,2,3], Hiroaki Kashima[4]

1. Disaster Prevention Research Institute, Kyoto University, Japan
2. Swansea University, United Kingdom
3. Typhoon Science and Technology Research Center, Yokohama National University, JP
4. Port and Airport Research Institute (PARI), Japan

2025/09/22

Introduction

■ Extreme Wave Height

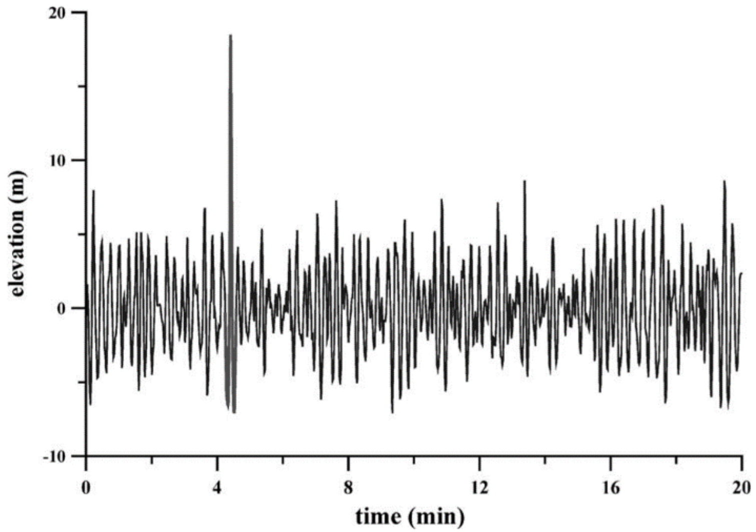
A special type of wave has been found in deep-water significantly deviating from Gaussian distribution.

Freak wave / rogue wave / extreme wave

■ One of the Generation Mechanism

Modulational instability from Benjamin (1967)

Janssen (2003) → an example of a non-resonant four-wave interaction in which the carrier wave is phase-locked with the sidebands.



Wave energy En in terms of wave steepness ε from Hamiltonian:

$$En = \varepsilon^2 En_2 + \varepsilon^3 En_3 + \varepsilon^4 En_4 + O(\varepsilon^5)$$

- Linear theory ($\varepsilon^2 En_2$)
- Three-wave interaction ($\varepsilon^3 En_3$)
- Four-wave interaction ($\varepsilon^4 En_4$)

Wave frequency ω may be written as:

$$\omega = \omega_1(\boldsymbol{k}) + a^2\omega_2(\boldsymbol{k}) + \cdots = \sqrt{gk}(1 + \frac{1}{2}\varepsilon^2 + \cdots)$$

Non-linear interaction from high-order and harmonics in the Zakharov equation:

$$\frac{\partial b_1}{\partial t} + \mathrm{i}\omega_1 b_1 = -\mathrm{i}\iiint_{-\infty}^{\infty} d\boldsymbol{k_2}\, d\boldsymbol{k_3} d\boldsymbol{k_4} T_{1,2,3,4} b_2^* b_3 b_4 \delta_{1+2-3-4}$$

Introduction

■ Problems in Two-dimensional wavefields

Complicated directional process:

wind forcing & bathymetry effect → Refraction & diffraction & … → Crossing seas

may trigger extreme events through multiple mechanisms

■ Research to date

Possible mechanisms:

- Linear superposition
- Second-order interactions (Fedele et al., 2016; Christou et al., 2009; Walker et al., 2004; McAllister et al., 2019; …)
- Third-order interactions (Gibson and Swan, 2007; Waseda et al., 2009; …)
- Nonlinear interactions between two systems (Davison et al., 2022; Trulsen et al., 2015; Gramstad et al., 2018; Luxmoore et al., 2019; ...)

■ What we want to do

Based on our previous numerical model, provide a more comprehensive discussion incorporating multiple external contributions:

- Wave crossing system
- Topography change
- Directional spreading
- Non-order interactions from NLS equation

Governing equation

■ Modified NLS equation in 2-D

This research aims to simulate the process water wave entering continental shelf, considering the effect of bottom topography change and wave interactions in 2-D directional random wavefield.

For an irrotational, inviscid and incompressible flow with free surface :

Non-linear boundary condition

- Laplace equation $$\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}+\frac{\partial^2 \Phi}{\partial z^2}=0$$
- Free surface equation $$\frac{\partial \eta}{\partial t}+\frac{\partial \Phi}{\partial x}\frac{\partial \eta}{\partial x}+\frac{\partial \Phi}{\partial y}\frac{\partial \eta}{\partial y}=\frac{\partial \Phi}{\partial z}, \qquad z=\zeta$$
- Bernoulli's equation $$\frac{\partial \Phi}{\partial t}+g\eta+\frac{1}{2}\left[\left(\frac{\partial \Phi}{\partial x}\right)^2+\left(\frac{\partial \Phi}{\partial y}\right)^2+\left(\frac{\partial \Phi}{\partial z}\right)^2\right]=0, \qquad z=\zeta$$
- Bottom boundary condition $$\frac{\partial \Phi}{\partial z}+\frac{\partial h}{\partial x}\frac{\partial \Phi}{\partial x}+\frac{\partial h}{\partial y}\frac{\partial \Phi}{\partial y}=0, \qquad z=-h(x)$$

Wave potential

$\Phi=\Phi(x,y,z,t)$

Surface elevation

$\eta=\eta(x,y,t)$

Water depth

$h=h(x)$

Governing equation

Suppose the bottom changes very mild, the depth varies slowly and explicit: $h'(x) = O(\varepsilon^2)$, $\tau = \varepsilon\left[\int^x \frac{dx}{c_g} - t\right]$, $\xi = \varepsilon^2 x$, $\zeta = \varepsilon y$

In third order, amplitude for first harmonic can be decided:

$$\mathrm{i}\lambda A + \mathrm{i}\frac{\partial A}{\partial \xi} + \gamma\frac{\partial^2 A}{\partial \zeta^2} + \lambda\frac{\partial^2 A}{\partial \tau^2} = \upsilon|A|^2 A$$

λ: shoaling effect depends on h

Surface elevation from envelope :

$$\eta = \varepsilon\mathrm{Re}\left[\frac{1}{2}\bar{A}\exp\big(\mathrm{i}(kx - \omega_0 t)\big)\right] + \varepsilon^2\mathrm{Re}\left[\frac{k\cosh kh}{8\sinh^3 kh}(2\cosh^2 kh + 1)\bar{A}^2\exp\big(2\mathrm{i}(kx - \omega_0 t)\big)\right]$$

$$k_0 = \sqrt{k_x^2 + k_y^2}$$

$$\theta_\zeta = \arctan\left(\frac{k_\zeta}{k_x}\right)$$

$k_x \approx k_0$ when small angle

If we consider a small oblique angle between bottom contour line and principal wave direction, then $k_x \approx k$ and we can simulate wave refraction:

$$\eta = \varepsilon\mathrm{Re}\left[\frac{1}{2}\bar{A}\exp\Big(\mathrm{i}\big(kx + k_\zeta y - \omega_0 t\big)\Big)\right] + \varepsilon^2\mathrm{Re}\left[\frac{k\cosh kh}{8\sinh^3 kh}(2\cosh^2 kh + 1)\bar{A}^2\exp\Big(2\mathrm{i}\big(kx + k_\zeta y - \omega_0 t\big)\Big)\right]$$

For two wave trains from similar initial condition with different principal direction, our numerical model could simulate their crossing case. Their interaction are neglected (smaller than 3rd order term) at small oblique angle.

$$\eta = \sum \eta_m, \qquad \eta_m = \varepsilon\mathrm{Re}\left[\frac{1}{2}\bar{A}\exp\big(\mathrm{i}(kx + k_m y - \omega_0 t)\big)\right] + \varepsilon^2\mathrm{Re}\left[\frac{k\cosh kh}{8\sinh^3 kh}(2\cosh^2 kh + 1)\bar{A}^2\exp\big(2\mathrm{i}(kx + k_m y - \omega_0 t)\big)\right]$$

Numerical Modeling

■ Initial Condition of 2-D Directional Wavefield

2D Gaussian shape directional spectral of Fourier amplitude:

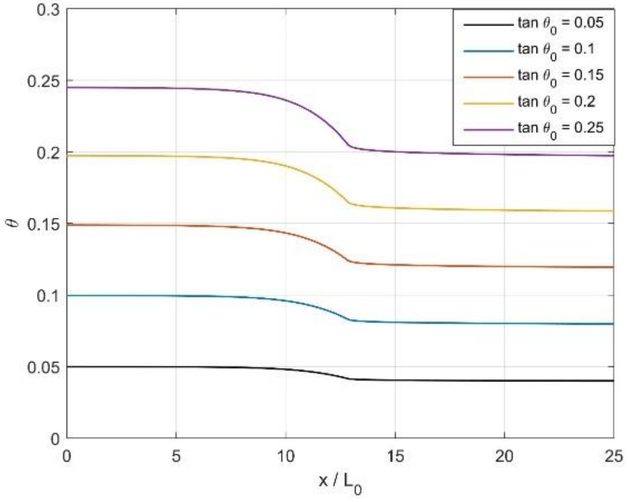
$$\ddot{A}(\omega_\tau, \xi_0, k_\zeta) = \ddot{A}(\omega_\tau, \xi_0, \theta_\zeta) = \frac{a}{2\pi\sigma_\omega\sigma_\theta}\exp\left\{-\frac{1}{2}\left[\left(\frac{\omega_\tau-\omega_0}{\sigma_\omega}\right)^2+\left(\frac{\theta_\zeta-\theta_0}{\sigma_\theta}\right)^2\right]+i\psi\right\}$$

ψ: Random phase

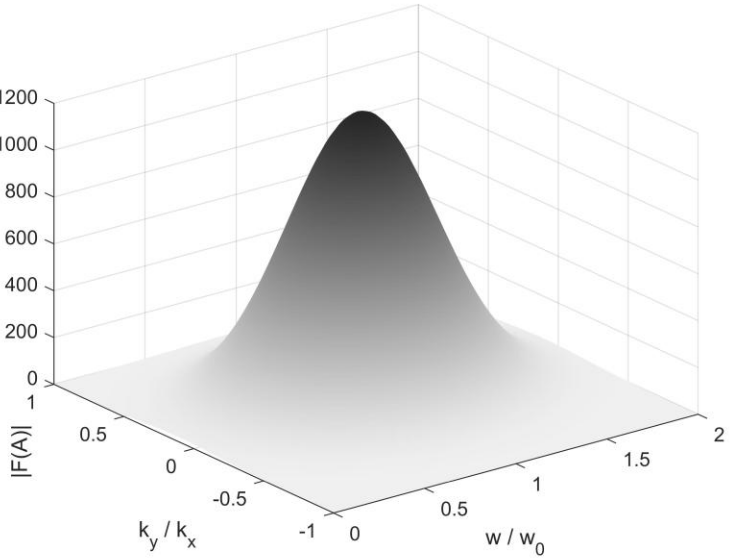
σ_θ: directional spread

θ_0: principal wave direction

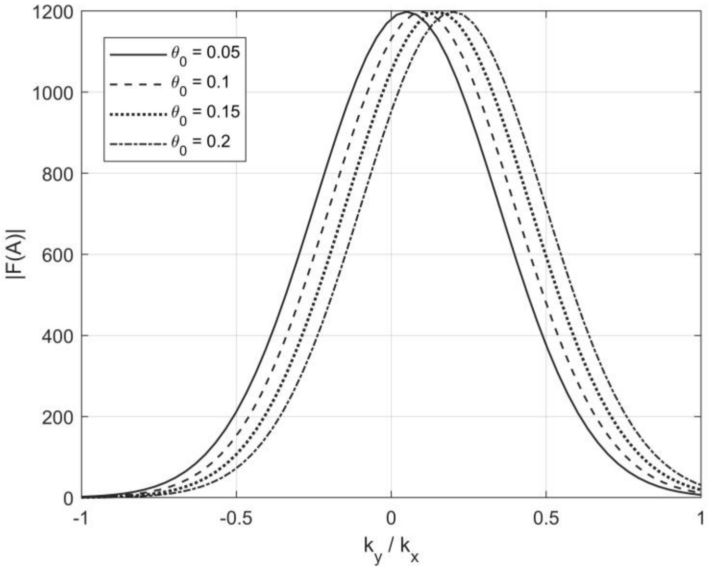
$\theta_\zeta = \arctan\left(\frac{k_\zeta}{k_0}\right)$



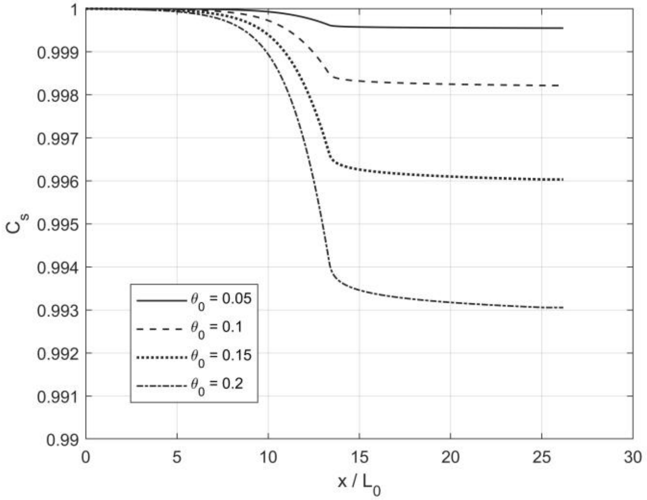
Wave refraction on a slope



Initial 2D spectral for frequency and direction



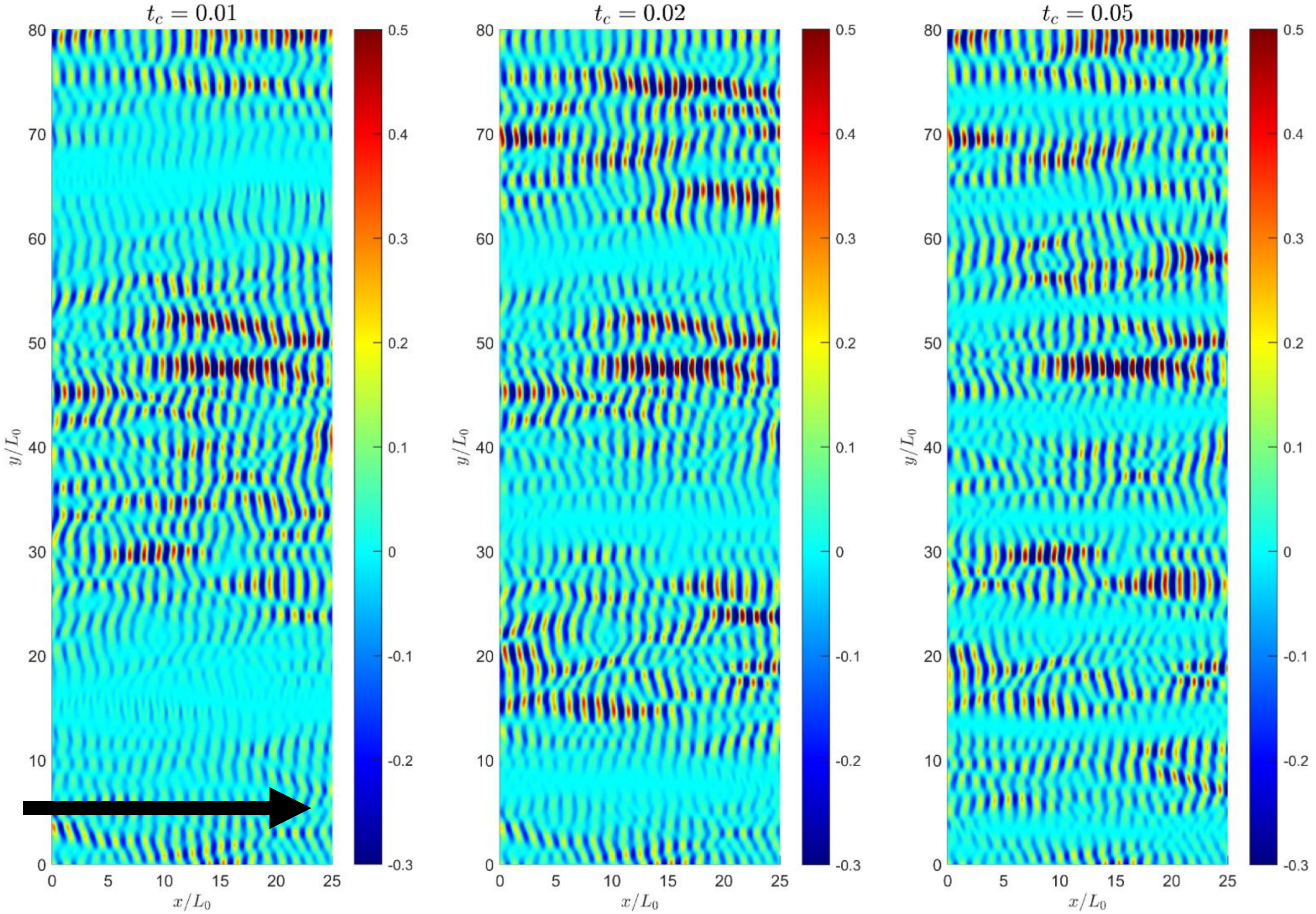
Peak shift from different θ_0



Error estimation from $k_x \approx k$ on a slope

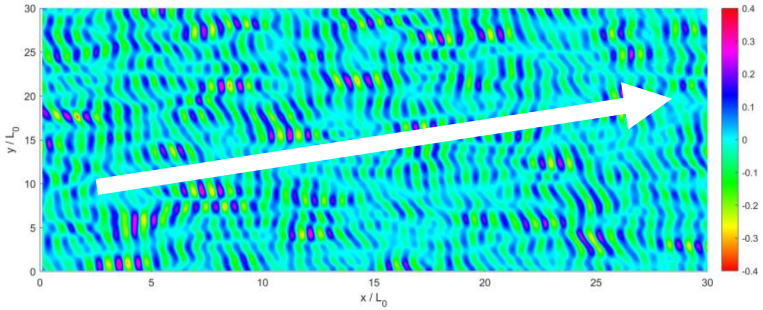
Numerical Modeling

■ **Surface elevation for a crossing system**

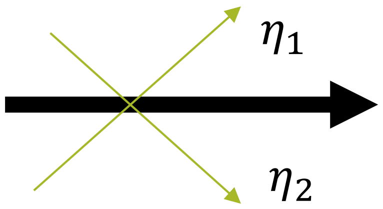


Single wave train

$\sigma_\theta = 0.3,\ \theta_0 = 0.1$



Total surface equals to their linear superposition



Crossing angle defined by

$$t_c = \tan\theta_{01} = -\tan\theta_{01}$$

Four-wave interaction: kurtosis μ_4

$$\mu_4 = \frac{\mathrm{Ex}(\eta - \bar{\eta})^4}{[\mathrm{Ex}(\eta - \bar{\eta})^2]^2}$$

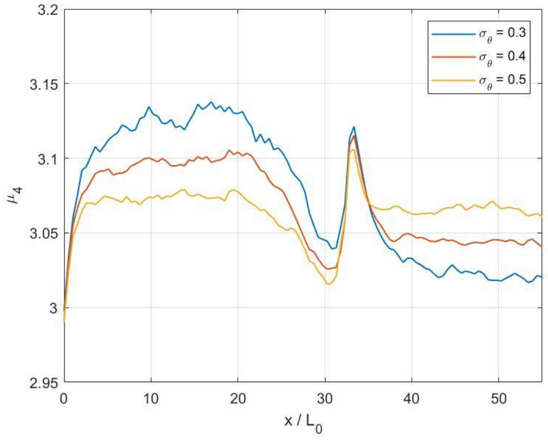
Three-wave interaction: skewness μ_3

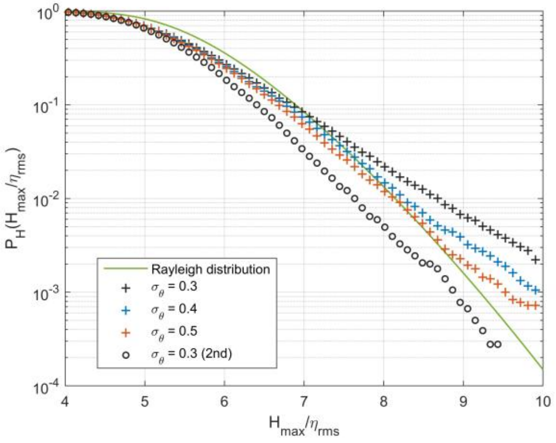
$$\mu_3 = \frac{\mathrm{Ex}(\eta - \bar{\eta})^3}{[\mathrm{Ex}(\eta - \bar{\eta})^2]^{\frac{3}{2}}}$$

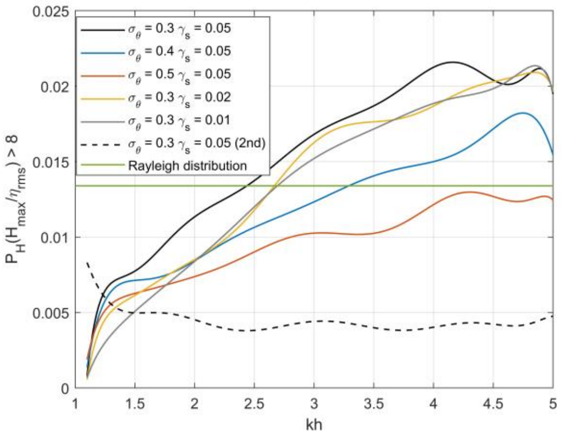
Simulation result

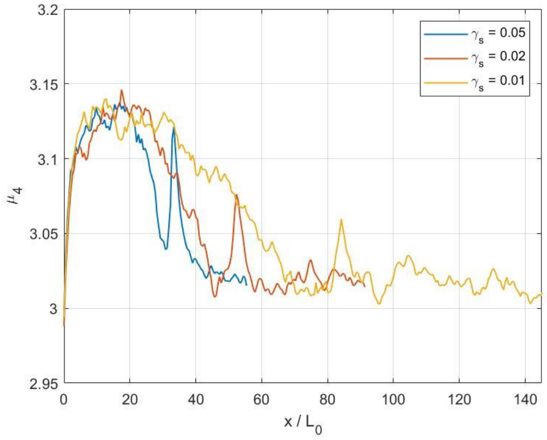
■ Result for single wave train in Monte Carlo simulation

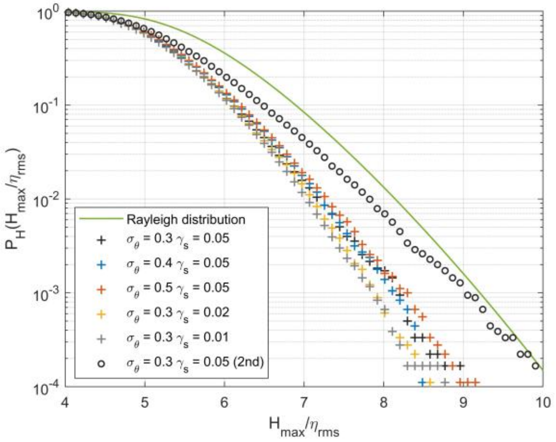
- ✓ High-order non-linear interaction
- ✓ Wave height distribution & probability of freak wave

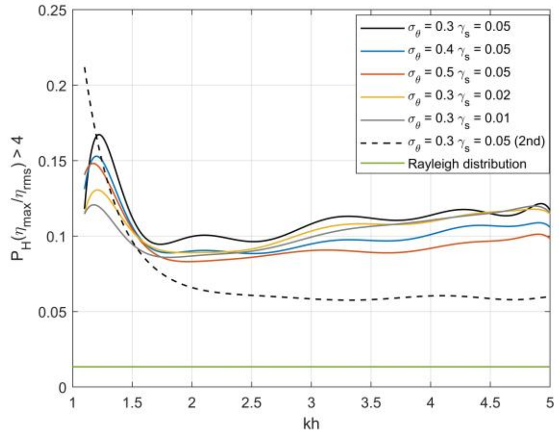


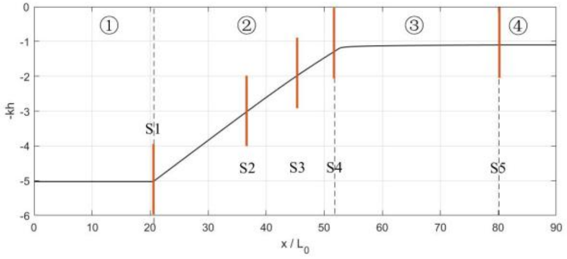












Bottom topography

Extreme wave height decided by:

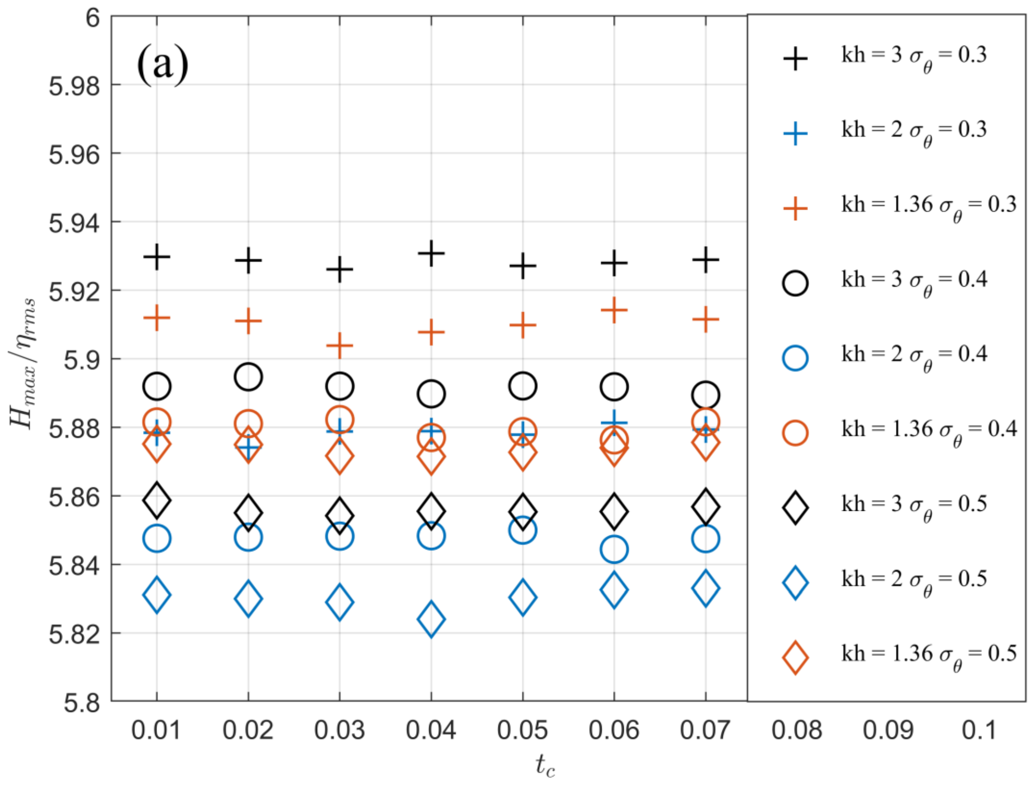
- **Initial condition:**
  Spectral bandwidth
  Directional dispersion
- **Boundary condition:**
  Bottom topography including slope angle & water depth

Simulation result

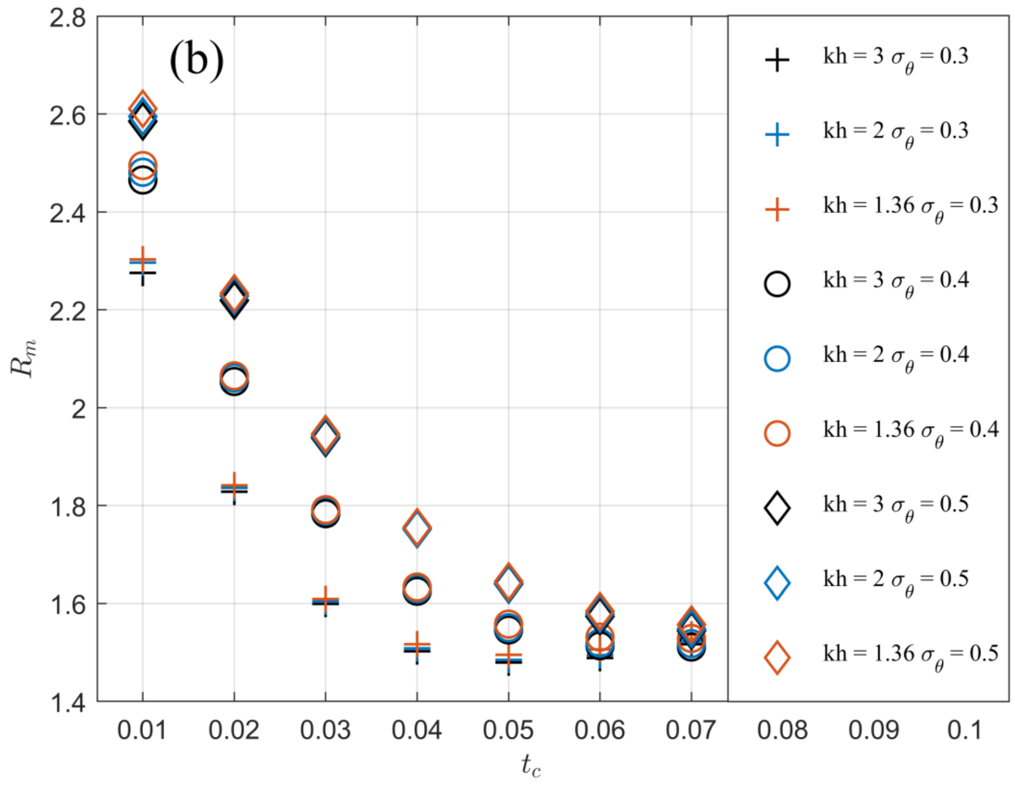
■ Maximum wave height & wave runlength

Wave runlength R_m: the mean value of numbers of consecutive waves exceeding the mean wave height

- Crossing angle contributes limited to Hmax
- The increase of crossing angle will decrease wave runlength, i.e., decrease the extreme events and energy concentration



Maximum wave height

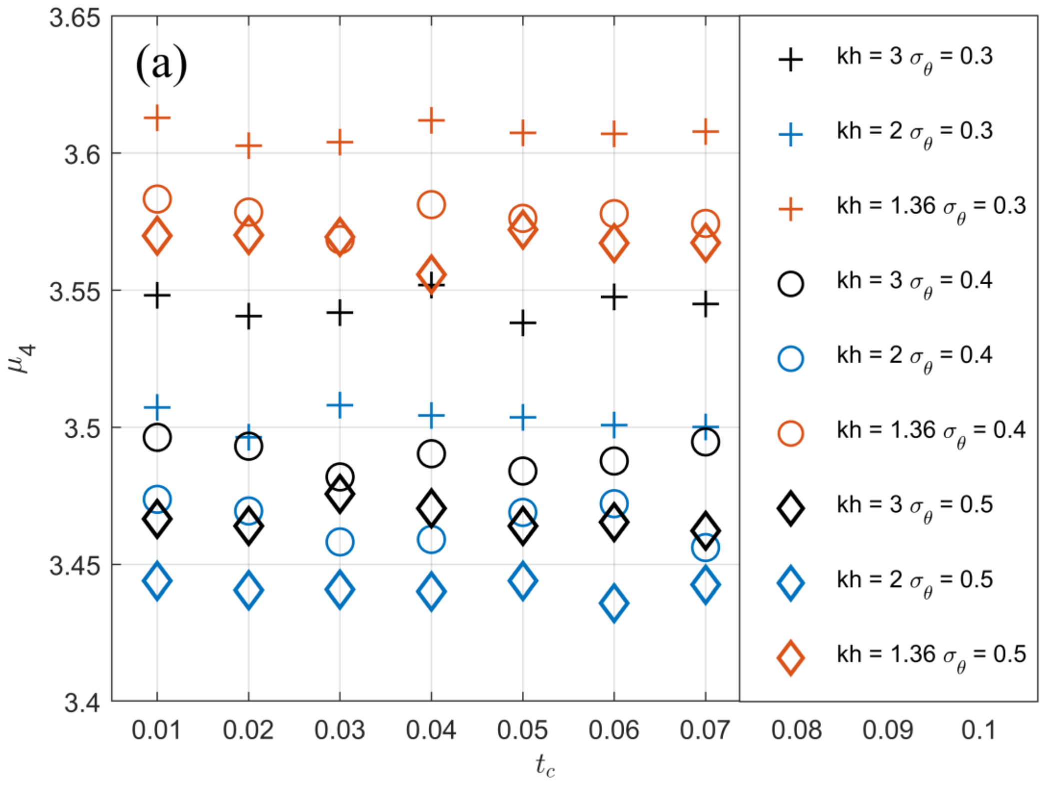


Wave runlength

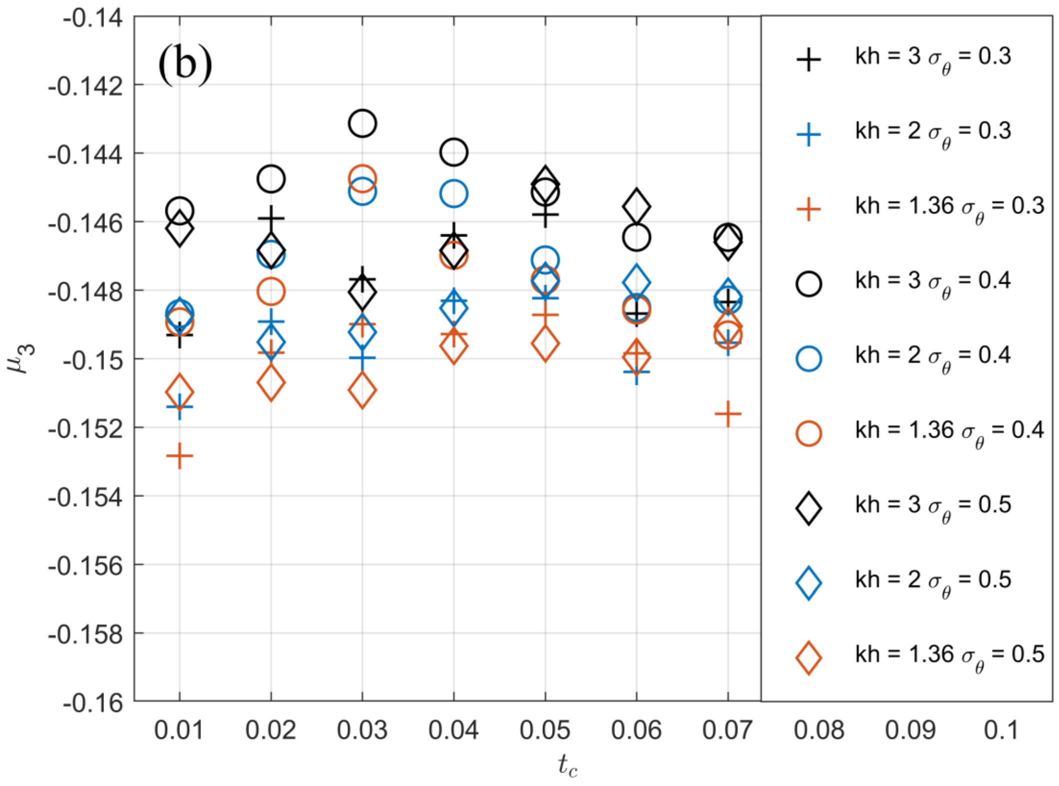
Simulation result

■ Kurtosis & skewness

- Crossing angle contributes limited to kurtosis
- We simulate with two symmetric wave trains, which limits the generality of skewness

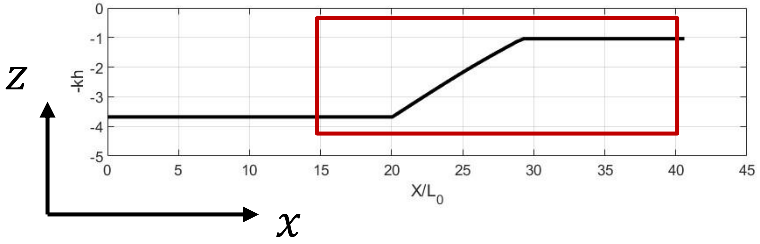


Kurtosis



Skewness

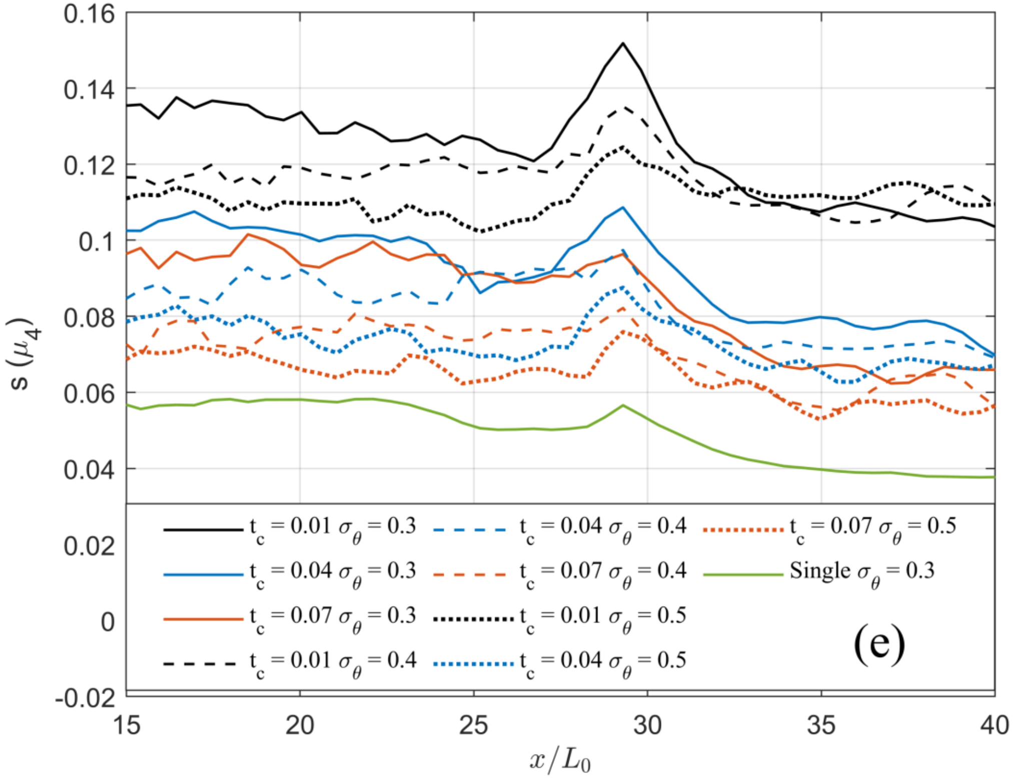
Simulation result



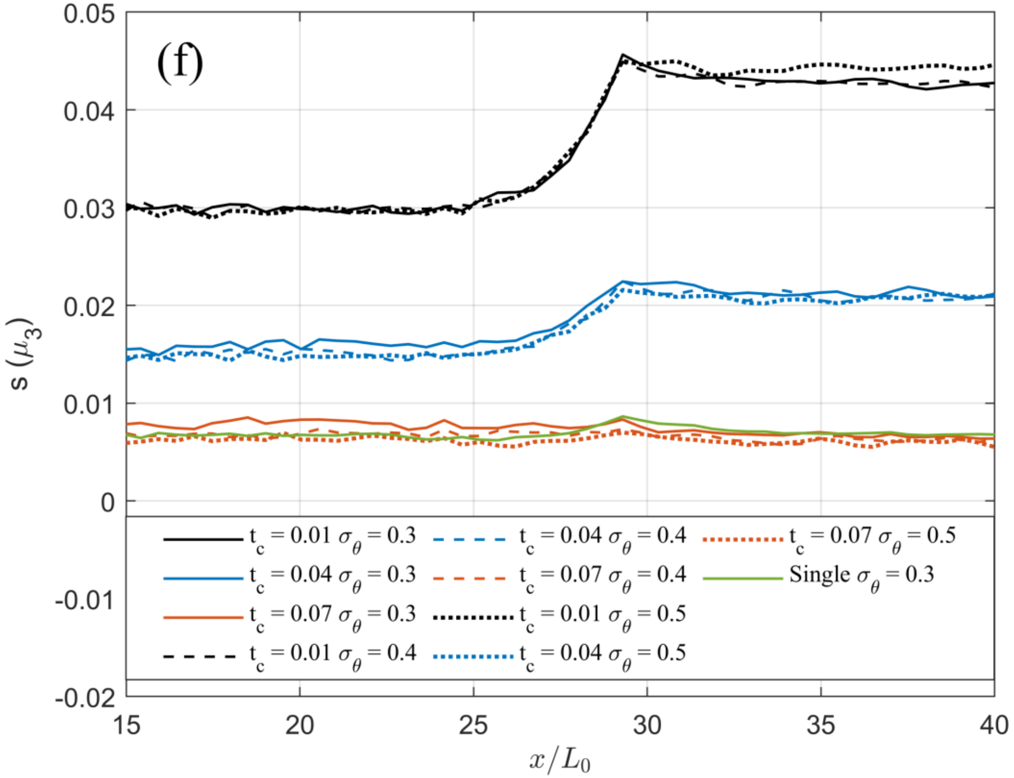
A sloping bottom

■ Standard deviations of kurtosis & skewness

- STD of kurtosis and skewness decrease notably, indicating a reduction in surface instability
- The increase of directional spreading σ_θ will make this process less noticeable



STD of kurtosis



STD of skewness

Summary

- Most study mainly discuss the crossing systems between 15~60 degree. We found even a small crossing angle contributes to the surface instability and wave morphology.
- Two-system wave surface gives a different evolution process with single system. The difference caused by crossing angles show up in standard deviation rather than averaged value——a higher order parameter in statistics.

Work in future

- Interactions between two coupling systems
- A wider range of crossing angle
- Two-dimensional bottom and wave-current

Thanks for your attention!