A Wind Wave Model Using Stabilized Finite Elements and Implicit/Explicit Time-Stepping

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Introduction

Spectral wind wave models such as WAVEWATCH III and SWAN continue to be applied in coastal applications. In these applications, utilizing unstructured meshes along with implicit time stepping in order to capture the irregular bathymetries of coastlines while avoiding restrictive CFL conditions can save computational cost without sacrificing accuracy. The finite element method is a numerical method that is appealing for this kind of situation since it allows for unstructured meshes, is capable of higher order approximations, and is backed by rigorous approximation theory. In this work, a finite element model for the Wave Action Balance Equation (WAE) is developed using the open-source FEniCSx framework. The model is applied to some test cases from the Office of Naval Research (ONR) Test Bed and compared to analytic solutions, lab data, and SWAN output.

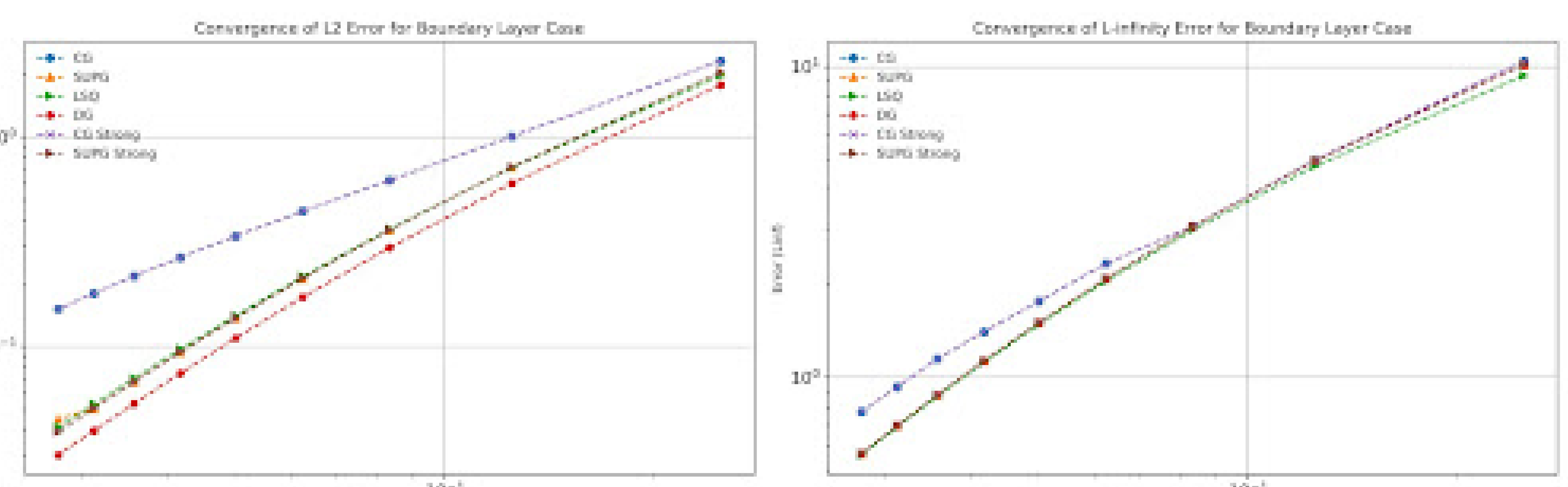

Figure: Convergence results in L^2 (left) and L^∞ norms with respect to h refinement

Methods

To model wave action with finite elements, we discretize a weak form of the WAE in $\Omega \subset (x, y, \sigma, \theta)$ along with a finite difference approximation in time. Since the WAE is an advection equation, stabilized finite elements techniques must be employed. The Streamline Upwind Petrov-Galerkin (SUPG), Least Squares, and Discontinuous Galerkin (DG) methods were all implemented for simplified domains $\Omega \subset (x, \sigma)$ but for the purposes of solving the full WAE, the SUPG method is employed, find $N \in U(\Omega, t)$:

$$\left(\frac{\partial N}{\partial t}, w\right)_\Omega - (\mathbf{c}N, \nabla w)_\Omega + (\mathbf{c}N \cdot \mathbf{n}, w)_{\Gamma_+} + \left(\frac{\partial N}{\partial t} + \nabla \cdot (\mathbf{c}N) - \frac{S}{\sigma}, \tau \mathbf{c} \cdot \nabla(w)\right)_{\Omega_e} = \left(\frac{S}{\sigma}, w\right)_\Omega \quad \forall w \in V(\Omega).$$

When source terms, S, are turned off, the time derivative is estimated with the second order Crank-Nicolson implicit approximation. When source terms are active, the discrete operator is split between advection and sources via the 2nd order Strang splitting scheme. The advection operator still uses the Crank-Nicolson scheme to advance in time while the source terms use an explicit second order Runge-Kutta scheme to advance in time. The model is implemented in Python using the FEniCSx library which has built-in parallelism via MPI. Furthermore, the model is capable of unstructured meshes in both geographic and spectral spaces.

The new wave model is applied to multiple test cases from the ONR test bed including shoaling, refraction, currents, and a wave breaking case. The shoaling, refraction, and breaking cases were tested with unstructured meshes in geographic space. The shoaling, refraction, and currents cases have analytic solutions to compare against while the wave breaking case is compared to both lab data and SWAN output.

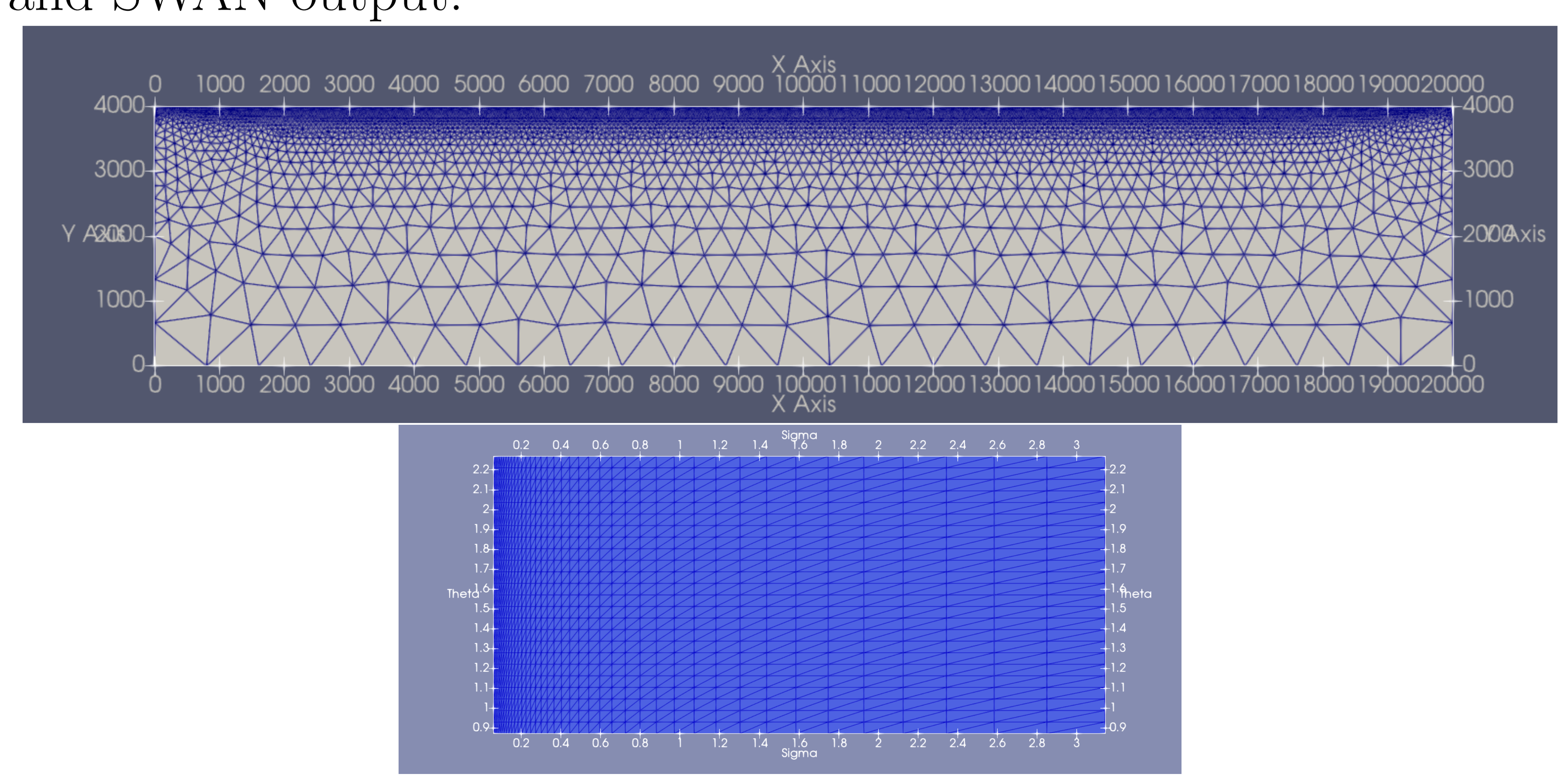

Figure: Unstructured meshes employed in test cases (geographic on top, spectral on bottom)

Results and Discussion

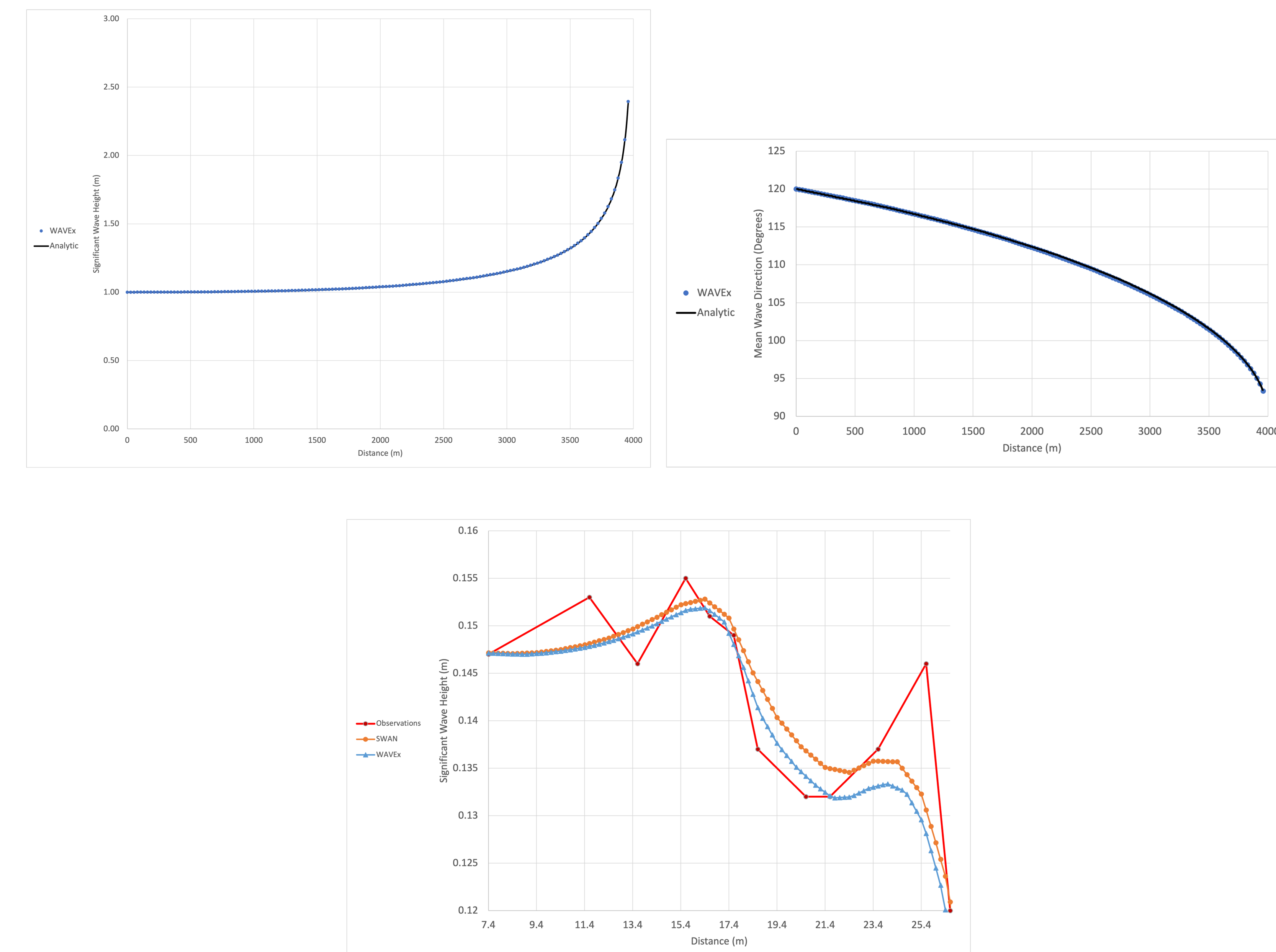

Figure: H_s and θ_{mean} of refraction case (top), and H_s of breaking case (bottom)

For the refraction case, the observed RMSE error in H_s is 0.00261 m and RMSE error in θ_{mean} is 0.119 degrees while the l^∞ error is 0.00865 m and 0.1946 degrees with respect to the analytic solution. For the wave breaking case the error in significant wave height with respect to the observations were 0.0064 m for WAVEx and 0.00556 m for SWAN while the l^∞ errors were 0.0179 m and 0.015 m respectively. The new wave model also showed robustness with respect to taking large time steps that violate the CFL condition.

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