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$h \sim 256 - 32$m

Solving 2D shallow water flow equations with subgrid approximation using a mixed-interpolation FE method

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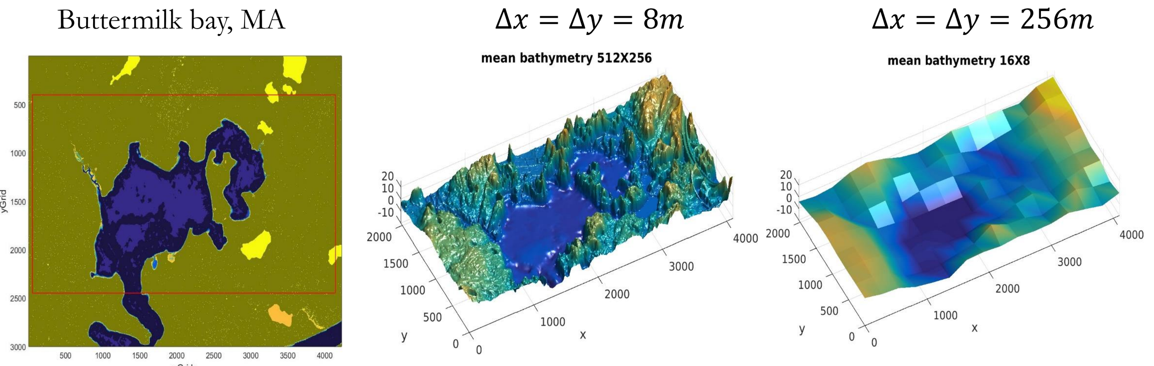
Introduction

Background

- Shallow Water Equations (SWE) are suitable for many real world applications, e.g. tide and storm surge modeling, river flooding, tsunami, etc.
- SWE are typically integrated using discretization methods, e.g. FD, FV, FEM, on fixed grids.
- Faithfully representation of a ground elevation using a sufficiently high resolution grid is important for model accuracy.



- A grid resolving complex topo/bathy in a large domain could consist of very high DOFs so that the computing time required to obtain results is prohibitively high.
- A tradeoff between accuracy and computing time is often made in practice.

Subgrid Approach

- Subgrid approaches offers a means to improve the level of accuracy of coarse-grid calculations.
- These approaches incorporate bulk influences of high-resolution ground elevation finer than the grid size in the model formulation.
- This study considers an approach of Kennedy et al., 2019 [1], which used formal averaging techniques to generate 'grid-scale' forms of the SWEs as follows:

$$\frac{\partial H}{\partial t} + \nabla \cdot (\boldsymbol{u}H) = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + f\widehat{\mathbf{k}} \times \boldsymbol{u} = -g\nabla\eta - \frac{\phi c_{M,f}}{H}|\boldsymbol{u}|\boldsymbol{u} - \frac{1}{\rho_o}\frac{\partial P_A}{\partial x} + \frac{\phi \rho_a c_d}{\rho_0 H}|\boldsymbol{W}|\boldsymbol{W}$$

where

η = surface elevation, $\boldsymbol{u} = (u,v)$ = grid-scale velocity

$$H(\boldsymbol{x},t) = \frac{1}{|A_G|}\int_{A_G} \max(0, \eta + b(\boldsymbol{x}'))d\boldsymbol{x}'.$$

$$\phi(\eta) = \frac{|A_W|}{|A_G|}, \quad A_W = \{\boldsymbol{x}' \in A_G \mid \eta + b(\boldsymbol{x}') > 0\}$$

$$c_{M,f} = \frac{1}{|A_W|}\frac{\left[\int_{A_G} \max(0,\eta + b(\boldsymbol{x}'))d\boldsymbol{x}'\right]^3}{\left[\int_{A_G}\max(0,\eta+b(\boldsymbol{x}'))^{\frac{3}{2}}\sqrt{1/c_f}d\boldsymbol{x}'\right]^2}$$

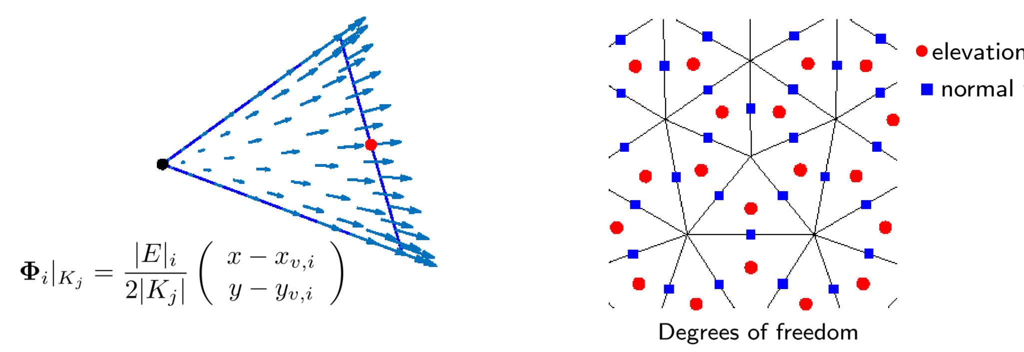
Discretization

Semi-implicit Galerkin FEM

- Semi-implicit θ-scheme for temporal discretization.
- The lowest order Raviart-Thomas (RT0) Galerkin FEM on a triangle mesh for spatial discretization.

$$\eta_h = \sum_{i=1}^{N_{el}} \eta_i(t)\mathcal{X}_i(\boldsymbol{x}), \ \mathcal{X}_i(\boldsymbol{x}) = \begin{cases} 1, & \boldsymbol{x} \in K_i \\ 0, & \text{otherwise}\end{cases}$$

$$\boldsymbol{u}_h = \sum_{i=1}^{Ned} u_{n,i}(t)\boldsymbol{\Phi}_i(\boldsymbol{x}), \quad \boldsymbol{\Phi}_i = \left[\phi_i^1 \quad \phi_i^2\right]^T, \boldsymbol{\Phi}_i \cdot \mathbf{n}_j = \delta_{ij}$$



$$\boldsymbol{\Phi}_i|_{K_j} = \frac{|E|_i}{2|K_j|}\begin{pmatrix} x - x_{v,i} \\ y - y_{v,i}\end{pmatrix}$$

Degrees of freedom

- Upwind discontinuous Galerkin for the advection term.
- This results in a mildly nonlinear system:

$$\frac{\boldsymbol{V}(\boldsymbol{\eta}^{n+1}) - \boldsymbol{V}(\boldsymbol{\eta}^n)}{\Delta t} + \theta\mathbf{C}\mathbf{u}^{n+1} = -(1-\theta)\mathbf{C}\mathbf{u}^n$$

$$[\mathbf{M} + \theta\Delta t\mathbf{M}_b]\mathbf{u}^{n+1} = g\Delta t\theta\mathbf{G}\boldsymbol{\eta}^{n+1} + \mathbf{r}$$

$$[\mathbf{M}]_{ij} = \textstyle\int_\Omega \boldsymbol{\Phi}_i\boldsymbol{\Phi}_j d\boldsymbol{x}; [\mathbf{G}]_{ij} = \int_\Omega \mathcal{X}_j\nabla\cdot\boldsymbol{\Phi}_i d\boldsymbol{x}$$

- RT0 FE method conserves mass locally and is free of a spurious oscillation mode in surface elevation.
- It is a first-order accurate method.

Numerical Results

Parabolic bowl problem

- Idealized test case with analytical solution of oscillatory flow with moving shorelines in a rotating basin of paraboloid shape.
- Initial conditions:

$$\boldsymbol{u}(\boldsymbol{x},0) = \frac{1}{2C_0}\left[f(\sqrt{1-C^2} - C_0)\right](-y,x)$$

$$H(\boldsymbol{x},0) = h_0\left[\frac{\sqrt{1-C^2}}{C_0} - \frac{r^2}{L^2}\left(\frac{1+C}{(C_0}\right)\right]$$

$$\eta(\boldsymbol{x},0) = H(\boldsymbol{x},0) - b(\boldsymbol{x})$$

$$C = \frac{(h_0+\zeta_0)^2 - h_0^2}{(h_0+\zeta_0)^2 + h_0^2}, \zeta_0 = \eta(\mathbf{0},0), \ C_0 = 1 - C \quad L = \sqrt{\frac{8gh_0}{\omega^2 - f^2}} \text{ for a given } \omega$$

- Test configurations:

$h_0 = 50$m, $\zeta_0 = 2$m,

$f = 1.03 \times 10^{-3}$, $\omega = 2\pi/12$ (hr[-1])

$\theta = 0.55$

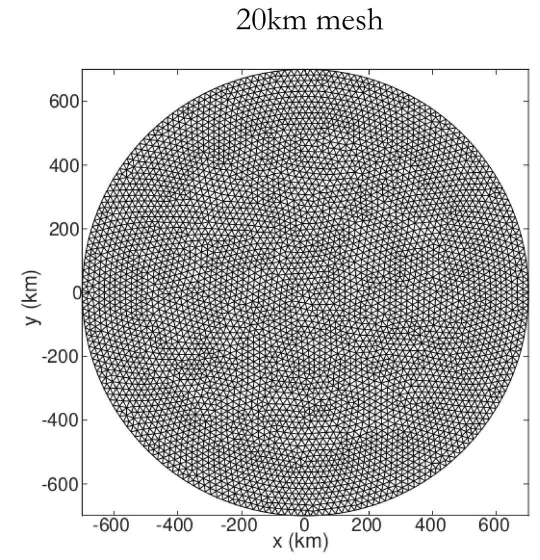
quasi-uniform meshes:

$h = 20, 10, 5$km

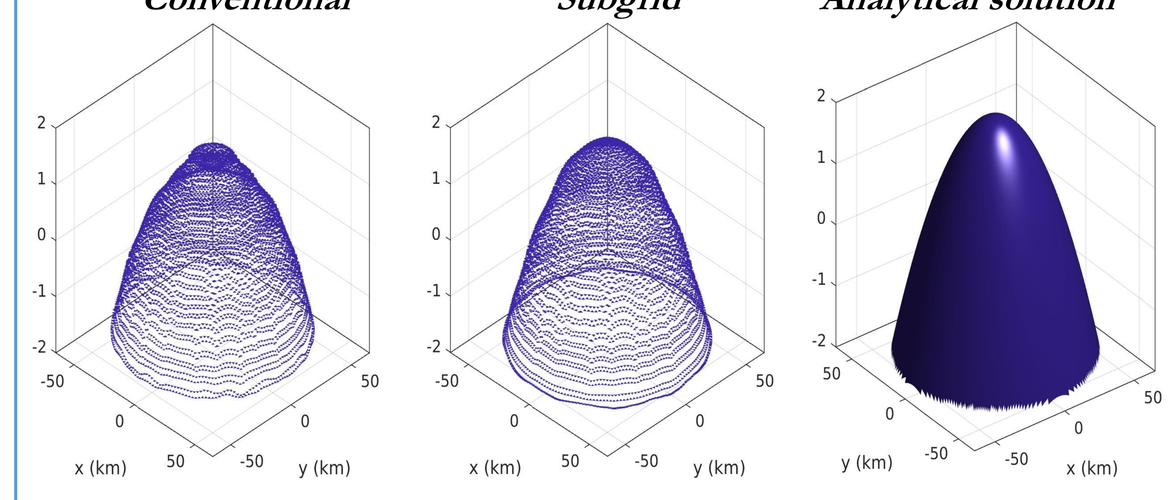
subgrid bathy resolution:

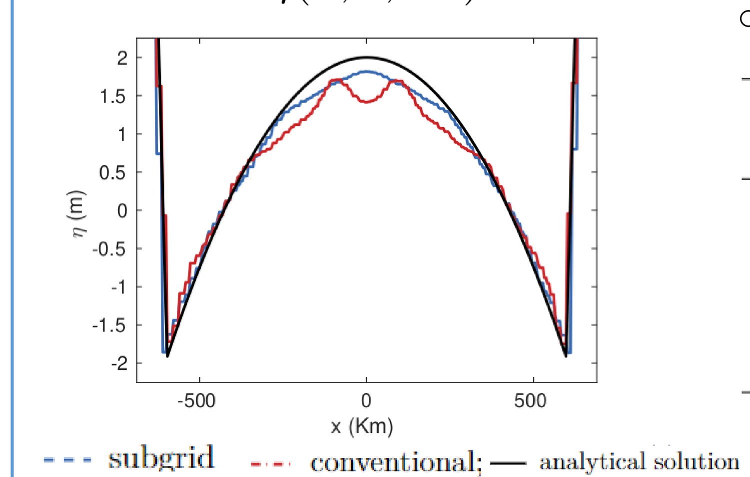
$h_s = 1.25$km

20km mesh



- Solution at $t = 6T$ on 20km mesh

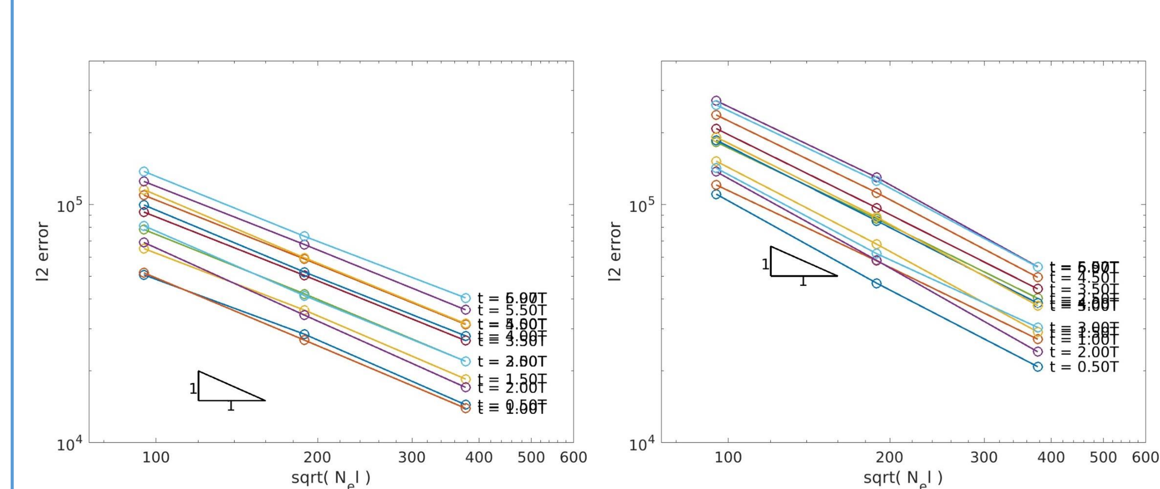




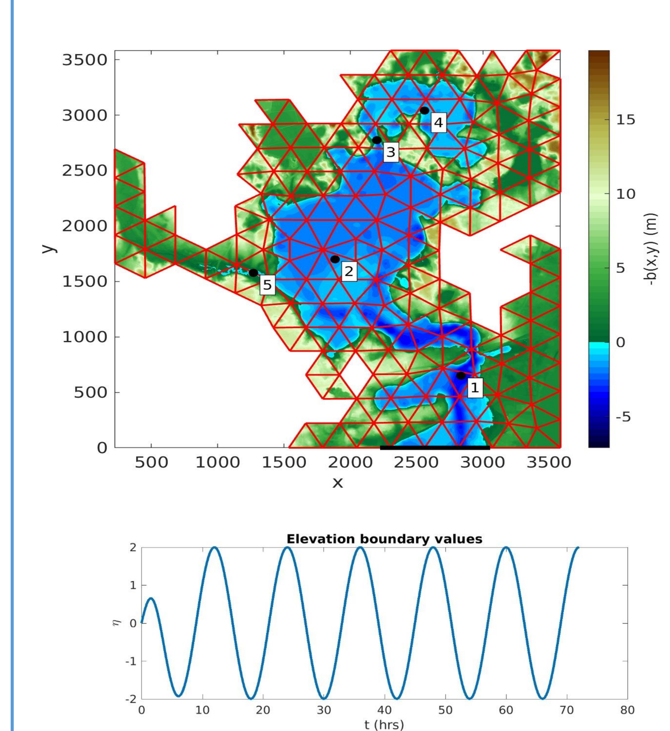
- Mass conservation

Time	$\frac{V(t)-V(0)}{V(0)}$ subgrid	$\frac{V(t)-V(0)}{V(0)}$ conventional
$1T$	4.00e-16	1.33e-16
$2T$	1.87e-15	2.67e-16
$3T$	6.67e-16	6.67e-16
$4T$	4.00e-16	8.00e-16
$5T$	4.00e-16	9.33e-16
$6T$	6.67e-16	9.33e-16

- L_2 convergence in surface elevation



Buttermilk bay



Test configuration

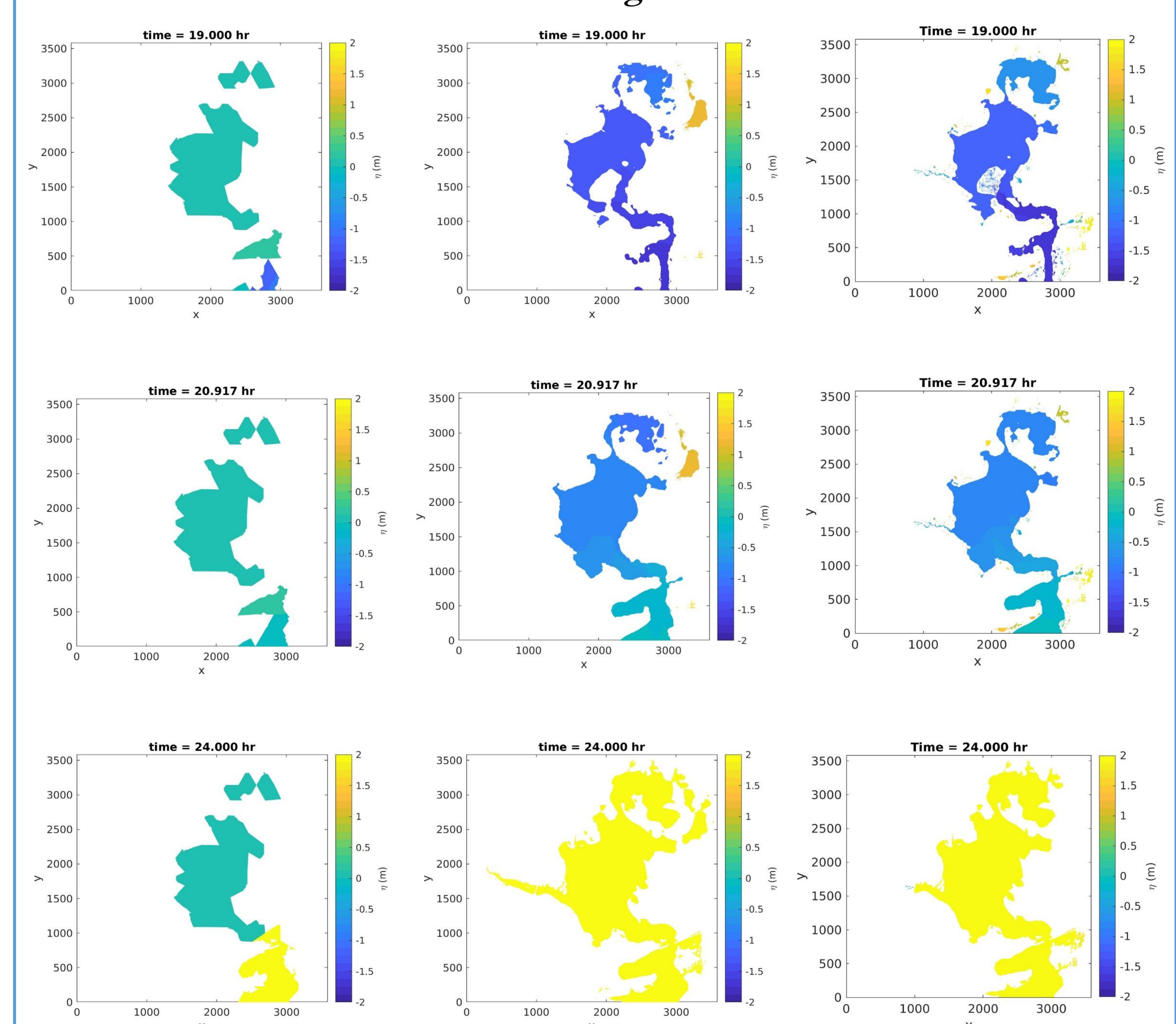
- Flow is driven by elevation boundary.
- Manning friction: $n = 0.02$
- Semi-implicit with θ=0.5
- Quasi uniform meshes:

  $h \sim 256$m-32m

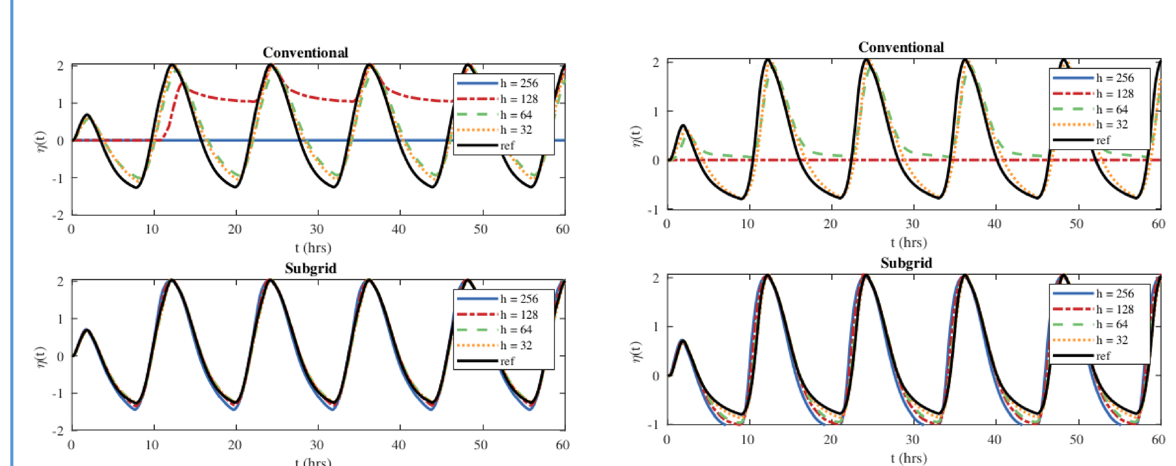
- Subgrid bathymetry resolution

  $h_s = 2$m

- Surface elevation



- Time series at stations



Summary

✓ An RT0 FEM is developed for the grid-scale SWE.
✓ Verification using idealized non-trivial test case with analytical solution.
✓ Validation using hurricane-induced storm surge processes with observation data

References

[1] Kennedy, A.B., Wirasaet, D., Begmohammadi, A., Sherman, T., Bolster, D. and Dietrich, J.C., 2019. Subgrid theory for storm surge modeling. *Ocean Modelling*, 144, p.101491.

Acknowledgments

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