

Inter-comparison study of methods for computing non-linear four-wave interactions in discrete spectral wave models

Gerbrant Ph. van Vledder^{1,2}
Noriaki Hashimoto³

¹*Delft University of Technology, Delft, The Netherlands, e: g.p.vanvledder@tudelft.nl*

²*BMT ARGOSS, Vollenhove, The Netherland*

³*Kyushu University, Fukuoka, Japan, e: hashimoto-n@civil.kyushu-u.ac.jp*

Abstract

This paper addresses the setup and first results of an inter-comparison study of methods for computing the non-linear four-wave interactions in discrete spectral models. The purpose of this study is to provide an objective framework to inter-compare computational methods which all try to solve the same complicated kinetic equation. A description is given of various theoretical and numerical approaches, each trying to solve this equation for a given discrete wave spectrum. For the inter-comparison a test strategy is proposed, consisting of simple static tests for a given spectral resolution and input spectrum, and dynamic tests in which the time- and/or space evolution of a wave spectrum is evaluated in conjunction with other source terms for wind input and dissipation. In this paper, static tests are carried out using two quasi-exact approaches from literature (WRT and RIAM) and two approximate approaches (DIA and SRIAM). The results of the inter-comparison reveal various differences that can be used to further improve the computational methods. It is pointed out that the proposed test strategy can also be used to objectively judge the performance of new computational methods.

1. Introduction

In the wind-wave modelling community it is generally accepted that non-linear four-wave interactions play an important role in the evolution of wind generated waves (Hasselmann et al., 1973; Young and Van Vledder, 1993). This understanding played a key role in the development of spectral wave-prediction models, finally resulting in the present day third-generation discrete spectral wave prediction models (WAMDI, 1988). This class of models is based on the action balance equation containing terms for the propagation of wave action in time and space, and terms describing the growth and decay of each wave component and the exchange of wave action between spectral wave components. .

Presently, a number of computational methods have been developed to exactly compute the non-linear transfer rate (S_{nl4}) for a given discrete wind wave spectrum (exactness only restricted by numerical method). This basic equation is often referred to as the Hasselmann (1962) or Zakharov (1968) equation. These methods differ in the way the delta-functions have been removed analytically from the original formulation and in the numerical treatment for solving the remaining integrals. Despite the fact that all of these methods are based on the same basic equation, no clear understanding exists whether these computational methods produce the same result for equal input. Because of the complicated structure of the basic equation, the full solution is not easily applicable in operational wave prediction methods like WAM (WAMDI, 1988), WaveWatch (Tolman, 1991) or SWAN (Booij et al., 1999). To achieve an

operationally feasible model, many approximate methods have been developed. As exact methods are being used as benchmark for the development of approximate methods, confidence in the exact methods is required.

The exact methods differ in many ways and unfortunately no objective statements can be made how these methods differ qualitatively and quantitatively from each other. Concerning exact solutions, three main lines of solution techniques have been described in literature. These are WRT (Webb, 1978), RIAM (Masuda, 1980) and the GQM (Lavrenov, 1992). Recently, a fourth method was presented at the 2013 WISE meeting due to Onorato et al. (2013), but its details are not yet published. The main differences between these methods exist in analytical transformations to make the basic equation suitable for numerical integration, but also in quadrature integration schemes, computational requirements and accuracy. This plethora of options makes it impossible to make objective claims which method is the best, all the more because no proper definition exists how to judge each method.

To provide clarity in the performance of these and other methods as well, a framework for an objective inter-comparison is presented in this paper. It is our intention that this inter-comparison study is carried out in the same spirit and in some aspects similar to the SWAMP study (SWAMP Group, 1982) for wave prediction models. It is expected that this inter-comparison study will also lead to a deeper insight into the inner workings of each method and to suggestions for improvements.

This inter-comparison should provide answers to a range of basic questions like:

- Do the methods solve the same equation?
- Are the solution techniques convergent with increasing resolution?
- Do the methods give the same answer for equal input?
- What is the sensitivity of each method to spectral resolution?
- What are their computational requirements in relation to their accuracy?
- Which assumptions are implicitly made about the spectral shape?

The inter-comparison is not aimed to carry out a computational method competition. In this sense there will be no winners or losers: each method may claim specific strong points or weaknesses (SWAMP group, 1982). It is our conviction that results of such an inter-comparison will be useful in pointing out future research in developing efficient and accurate methods for the computation of quadruplet interactions in discrete spectral wave models.

This paper presents results of two exact methods (WRT and RIAM) and two widely used approximate methods (DIA and SRIAM) for a number of static tests. It is our intention to extend the inter-comparison also to other exact and approximate computational methods, such as neural networks (Krasnopolsky et al., 2002, 2003; Wahle et al., 2009), Advanced Dominant Interaction method (Perrie et al., 2010), TSA method of Perrie and Resio (2008) and Perrie et al. (2013), or the GMD of Tolman (2013). In addition, the tests should also be extended to dynamic tests in which each computational method is implemented in a host numerical wave model, followed by 1D- and 2D non-stationary tests.

The layout of this paper is as follows. In Section 2 the overall test strategy is outlined. A brief description of the WRT, RIAM and SRIAM methods are provided in Section 3. The setup of the inter-comparison study is described in Section 4. A selection of results of the inter-comparison is presented in Section 5 for a limited number of test conditions. A discussion and conclusions are presented in the Sections 6 and 7.

2. Test strategy

2.1. General considerations

The basic principle of the inter-comparison study is to apply each computational method to a discrete wave spectrum and to inter-compare the resulting transfer rates. In the ideal case the results should be equal to each other for the exact methods and close to each other for the approximate methods. Preliminary computations already showed differences in results for the exact methods, indicating that differences in approach can lead to different answers. Understanding the cause of these differences is of interest to judge the sensitivities and limitations of each method, but also to point to improvements in each computational method.

The test strategy can be divided into static tests and dynamics tests. In a static test a discrete spectrum is used as input for a computational method and the resulting transfer rate is compared with other results. This strategy is often used in the development of new computational methods, often restricted to (smooth) academic JONSWAP spectra. Although this is a necessary requirement, it is a too limited approach as it may hide inherent sensitivities and assumptions, especially regarding the smoothness of the input spectrum, or those regarding the spectral resolution and extent of the spectral grid. In view of the cubic nature of the source term for quadruplet interactions small local perturbations in spectral shape may cause a strong response that may be missed in certain computational methods. In addition, hardly noticeable deviations from the exact solution may accumulate to significant errors when repeatedly applied to a wave spectrum in a dynamic test, which may ultimately lead to instable results.

In a dynamic test a computational method is cast into a subroutine and implemented in a host wave model together with deep water source terms for wind input and whitecapping dissipation, and optionally also with shallow water source terms for wave breaking and bottom friction. Subsequently, this wave model is integrated with respect to time and/or space to determine the evolution of the wave spectrum for a range of conditions. Dynamical tests are essential to judge whether a computational method for the quadruplet interactions produces stable results, proper spectral shapes, or realistic integral wave parameters, when carried out on the same computer platform.

The quality of a computational method for the quadruplet interactions can be judged in various ways. The first is related to the question whether a method is able to reproduce known transfer rates. This question can be answered by performing static tests. The second way is related to the question whether a computational method as implemented in a wave model, produces proper wave model results, often expressed in terms of statistics of integral wave parameters or shapes of wave spectra or of the non-linear source term. The latter aspect is relevant as the quadruplet source term is usually used in conjunction with other source terms. As the other source terms should be considered as approximations to the physical processes they represent, errors in the quadruplet source term may be acceptable, as long as the combined

action of these source terms (source term package) produces acceptable results (Tolman and Grumbine, 2013).

As outlined in the introduction, an exact evaluation of the transfer integral is computationally expensive and not feasible for implementation in operational wave models. For this reason, approximate methods have been or are still being developed at the expense of accuracy. It is therefore important to find a proper balance between computational requirements and accuracy of each computational method, either for an individual spectrum or for overall model performance. See Van Vledder (2012) for a discussion on the balance between quality and performance of numerical wave models. This inter-comparison study may help in objectively assessing the quality and performance of computational methods for quadruplet interactions.

A common theme in the development of computational methods is the occurrence of improperly substantiated claims about the effectiveness of each method. Such claims are often based on a limited number of tests, or based on non-representative academic situations, e.g. smooth input spectra. Another obstacle for an objective inter-comparison is that the outcome of tests depends on the programming language, compiler (settings) and hardware configuration. The dynamic tests proposed in this paper, describe a way to actually make such an objective inter-comparison when run on the same computer platform.

The key components of the inter-comparison study are as follows:

- Description of computational methods;
- Static tests: inter-comparison of computational results for pre-defined discrete test spectra;
- Objective determination of computational requirements;
- Implementation and application of S_{nl4} methods in a 1D model and evaluation of growth curves;
- Application of S_{nl4} methods in a 2D model applied to academic and realistic field situations.

2.2. Description of computational methods

To interpret differences in results and to explain possibly surprising results a good understanding of each computational method is required. As mentioned in Section 1, at least three exact methods have been developed which are all based on the same basic equation. The basic six-fold integral equation contains two delta functions (one in terms of frequencies and one in terms of wave number vectors). Different analytical transformations have been applied to eliminate these delta functions. The resulting equations for each method contain a certain mix of one- and two-dimensional integrals which need to be integrated numerically. Here lies the origin of differences in computational results as each method makes certain choices in e.g. spectral resolution (both internally and externally), treatment of singularities, handling of the parametric spectral tail, interpolation techniques, filtering out less significant contributions, handling of symmetries, quadrature methods and step sizes. The consequences of these choices on the resulting transfer rate may depend on the smoothness of the input spectrum, the chosen quadrature method to solve the integrals and the spectral resolution.

Other aspects that are relevant for the inter-comparison are the choice of the coupling coefficient, fulfilment of conservation laws, the handling of pre-computed results, (test) options to steer the inner workings of a computational method, memory management, parallelisation and its computational efficiency.

2.3. Static tests

The first step in any inter-comparison is to apply each computational method to a discrete input spectrum for a certain water depth. The latter statement already expresses three aspects that need attention in the inter-comparison. Firstly, the discretisation of the spectrum in frequency-direction (or wave number – direction) space influences the result. Secondly, the shape of the two-dimensional spectrum may reveal weaknesses in a computational method. Thirdly, most methods have originally been derived for deep water, but their extension to shallow water is not always trivial. In addition, certain computational methods may have internal switches to steer e.g. the accuracy for the evaluation of internal integrals. All these aspects need to be addressed in the static tests

Spectral resolution and spectral tail

Like any numerical method the spectral resolution will influence the results. The following tests are proposed:

- Spectral resolution in frequency (wave number) and direction space of the input spectrum. Of special interest is the question whether the result converges with increasing spectral resolution.
- The extent of the frequency (wave number) grid to both lower and higher frequencies. As shown by Van Vledder (2006) the upper frequency should be at least 6 times the peak frequency of the input spectrum.
- The power of the spectral tail may influence the computational results, especially when the higher discrete frequency of the input spectrum is relatively close to the peak of the input spectrum.

Spectral shape

Variations in the shape of the input spectrum have a strong influence on the resulting transfer rate and certain variations can be used to test sensitivities and conservation laws. The following variations are proposed:

- A benchmark spectrum based on a JONSWAP spectrum with $f_p=0.2$ Hz, peak enhancement factor $\gamma=3.3$ and a directional spreading function according to $D(\theta)=\cos^{2s}(\theta/2)$ with $s=1$ and assuming deep water. A required result of this test should be symmetry of the solution.
- Peakedness of the frequency spectrum. It is proposed to apply JONSWAP spectra with different peak enhancement factors of $\gamma=1, 3.3, 7$ and 20 .
- Directional spreading. It is proposed to apply $\cos^{2s}(\theta/2)$ with s varying from 1 (broad spectrum), 2, 5, 10 and 20 (narrow spectrum).
- Shift of peak frequency to test theoretical scaling laws. Both a shift to another discrete peak frequency as to an intermediate frequency should be included;

- Doubling of the action density to test the cubic nature of this source term;
- Shift of mean direction to test rotational symmetry. Both a shift over an integer number of $\Delta\theta$'s as well as an inter-mediate value.
- Skewed spectrum, i.e. the frequency dependent mean direction of the input spectrum changes over 30 and 60 degrees between $f=f_p$ to $f=2f_p$.
- Perturbed spectrum in which the spectral density at $f=2f_p$ is halved to create an omni-directional gap in the wave spectrum (cf. Resio and Perrie, 1991 & Young and Van Vledder, 1993).
- Theoretical spectra described in literature (cf. Dungey and Hui, 1979, Hasselmann and Hasselmann, 1981; Hasselmann et al., 1985; Tanaka 2007) for which solutions have been presented.

Internal settings

Each computational method contains a number of internal choices affecting the computational result. These choices may relate to the quadrature method, the spacing and number of points for the evaluation of internal 1d-integral (as in the WRT method), or the filtering out of certain parts of the integration space. The list of internal choices may vary per computational method.

Van Vledder (2006) lists a number of internal switches for his implementation of the WRT method:

- Quadrature method for evaluation 1D-integrals over each locus in wave number space (e.g. Trapezoid rule, Simpson rule or Gauss-Legendre rule);
- Number and distribution of points on each locus (partly related to previous point);
- Interpolation method to obtain action density at each interacting wave number vector;
- Filtering options to omit contributions to the transfer integral for wave numbers that are well separated from each other in wave number space.
- Length of the lines in wave number space to solve the 'special case' when $k_1=k_3$ (See van Vledder, 2006) for details. It is noted that this part of the interactions only affect directional properties of the transfer integral.

Shallow water

Most computational methods are able to compute the non-linear transfer rate for wave spectrum on deep and shallow water. As shown by Herterich and Hasselmann (1980) and Van Vledder (2006) the shape of the resulting transfer integral changes with decreasing water depth. Assuming that all computational methods apply the same depth dependent coupling coefficient the following tests should be carried out:

- Deep water, and water depths such that $k_p d$ takes the values 10, 5, 1 and 0.5, with k_p the wave number at the peak frequency (i.e. not equal to the peak wave number when spectrum formulated in k-space) and d water depth.

2.4. Dynamic tests

As each computational method for evaluating the quadruplet interactions will be part of an academic or operational wave model, its performance in conjunction with other source terms should also be assessed. In this way the stability of integration can be established but also whether the computational method produces realistic results in terms of spectral shape and integral wave parameter values.

To carry out an objective inter-comparison it is required that each computational method should be available as a subroutine such that it can easily be plugged into a host wave model with a fixed set of source terms for wind input and whitecapping.

The dynamic tests should be carried in a number of steps with increasing complexity according to:

- Growth curve analysis for 1D- computations with different wind speeds and water depth. The analysis should include the comparison of integral wave parameters as well as a comparison of spectral shapes.
- Application to academic 2D-cases, e.g. a selection of those described by the SWAMP group (1982).
- Application in an operational model for field situations, e.g. Lake George, Australia or Lake Sloten, the Netherlands (Bottema and Van Vledder, 2009), and those applied by Tolman and Grumbine (2013).

A side result of the dynamics tests is that objective information can be obtained about the computational requirements of each computational method in an operational environment. This evaluation may include the effect of choices in the actual implementation of each computational method. The dynamic test will also shed light on the computational requirements and achieved accuracy of each method and host model.

3. Computation of non-linear four-wave interactions

3.1. Basic equations

The first description of the energy transfer between four different wave components was given by Phillips (1960). His theory of non-linear four-wave interactions was further extended to a random sea independently by Hasselmann (1962) and Zakharov (1968), and the resulting six-fold integral is known as the Boltzmann integral or kinetic equation, respectively. This integral describes the rate of change of wave action density n_i for the wave number vector \mathbf{k}_1 due to resonant interactions with three other wave number vectors \mathbf{k}_2 , \mathbf{k}_3 and \mathbf{k}_4 according to:

$$\begin{aligned} \frac{\partial n_1}{\partial t} = & \iiint G(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \times \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \times \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \\ & \times [n_1 n_2 (n_3 + n_4) - (n_1 + n_2) n_3 n_4] d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4 \end{aligned} \quad (3.1)$$

in which the δ -functions reflect the resonance conditions which also ensure conservation wave energy, wave action and wave momentum. The corresponding radian frequencies ω_i are coupled with the wave numbers k_i via the linear dispersion relation. The term G is the coupling coefficient which is a complicated function of the four wave number vectors \mathbf{k}_1 , \mathbf{k}_2 , \mathbf{k}_3 and \mathbf{k}_4 and depth. Expressions for this term in deep water were provided by Hasselmann (1962) and simplified by Webb (1978) and in a slightly corrected form by Dungey and Hui (1979). A shallow water version of the coupling coefficient was given by Herterich and Hasselmann (1980).

3.2. Analytical methods

The mathematical structure of the Boltzmann integral (3.1) is difficult to cast in a computational method as all kinds of advanced numerical techniques need to be applied to handle the δ -functions. However, it can be rewritten in a more manageable form by analytically integrating over the δ -functions. Integrating over the δ -functions effectively reduces the six-fold integral into a three-fold integral, although the integration space is a multi-dimensional manifold in wave number space. Further, this space has to satisfy the resonance conditions or, equivalently, the conservation laws.

Analytical methods were developed by various researchers, each making different choices in their analytical transformations. Three main methods exist, which mainly differ in the choice of transformation variables to eliminate the δ -functions and to handle singularities in the integration domain. The first method is due to Webb (1978), whose method describes the rate of change of action density at wave number vector \mathbf{k}_1 as a function of wave number vectors \mathbf{k}_1 and \mathbf{k}_3 . The second method is due to Masuda (1980) whose basic transformation variables are the wave number vectors \mathbf{k}_3 and \mathbf{k}_4 . The third method is due to Lavrenov (2001) who introduced Gaussian quadrature formulas adapted to these singularities arising from the transformations.

Each of these analytical methods has been cast into a numerical solution technique. The Webb (1978) approach has been cast in the so-named WRT-method by Tracy and Resio (1982). Resio and Perrie (1991) extended the WRT method to shallow water. Van Vledder (2006) made an operational version of the WRT method, which is now implemented in various third-generation wave models as an alternative to the DIA method. The Masuda (1980) method has been cast in the RIAM (Research Institute for Applied Mathematics) method, to which Komatsu and Masuda (1986) and Hashimoto et al. (1998) have contributed. The Lavrenov (2001) method has been cast into an operational code by Gagnaire-Renou et al. (2010) and is known as the Gaussian Quadrature Method (GQM). These three quasi-exact methods are also denoted as Xnl, derived from eXact Non-Linear transfer.

In this paper results are presented of computational results of the WRT, RIAM, SRIAM and DIA methods. Descriptions of these methods are repeated below, except for the DIA for which the reader is referred to Hasselmann et al. (1985), Van Vledder (2012) and Tolman (2013).

3.3. The WRT method

Following Resio and Perrie (1991) and Van Vledder (2006, 2009) the non-linear transfer rate or rate of change of the action density n_1 at wave number k_1 can be written as:

$$\frac{\partial n_1}{\partial t} = \iint k_3 dk_3 d\theta_3 T(\mathbf{k}_1, \mathbf{k}_3) \quad (3.2)$$

in which the transfer function T is given by:

$$T(\mathbf{k}_1, \mathbf{k}_3) = \iint d\mathbf{k}_2 d\mathbf{k}_4 \times G \times \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \times \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \times [n_1 n_3 (n_4 - n_2) + n_2 n_4 (n_3 - n_1)] \quad (3.3)$$

The term G is the coupling coefficient, which is a function of all four wave numbers involved in an interaction. The δ -functions reflect the resonance conditions, or equivalently, they ensure the conservation of energy, action and momentum. As described in Van Vledder (2006) removing the δ -functions in (3.2) leads to the following expression for the function T :

$$T(\mathbf{k}_1, \mathbf{k}_3) = \oint_s ds \times G \times J \times N_{1,2,3,4}. \quad (3.4)$$

The function T consists of a line integral over a closed locus in wave number space and where G , J and $N_{1,2,3,4}$ are functions of the locus coordinate s . Note that for $k_1=k_3$, the locus consist of a straight line in wave number space. The locus can be considered as the solution in wave number space of the resonance conditions for a given combination of the wave numbers \mathbf{k}_1 and \mathbf{k}_3 . In (3.4) J is the Jacobian term given by:

$$J = \left| \mathbf{c}_{g,2} - \mathbf{c}_{g,4} \right|^{-1} \quad (3.5)$$

The term $N_{1,2,3,4}$ is the product term of action densities

$$N_{1,2,3,4} = n_1 n_3 (n_4 - n_2) + n_2 n_4 (n_3 - n_1) \quad (3.6)$$

Details about the determination of the locus and the evaluation of (3.4) can be found in Van Vledder (2006). For this paper it is relevant to realize that (3.4) consists of a 1D-line integral through wave number space for which the number and spacing of the abscissa affects the accuracy of the function T . In addition, various interpolation techniques can be applied to infer the action density at discrete points on the locus.

3.4. The RIAM and SRIAM method

The RIAM method

The (exact) RIAM method was developed from the initial work of Masuda (1980) who derived a reduction of the six-fold Boltzmann integral to a three-fold integral by taking the independent parameters $(\theta_1, \omega_3, \theta_3)$ as expressed by the following equation;

$$\frac{\partial \phi_4}{\partial t} = (2\omega_4^{23}) \int_0^\pi d\tilde{\theta}_3 \int_0^\infty d\tilde{\Omega} \int_0^\pi d\tilde{\theta}_1 \sum_{\pm} \sum_{\pm} \sum_{\pm} (8\tilde{\omega}_1^3 \tilde{\omega}_3^4 \tilde{G} S^{-1}) \{n_1 n_2 (n_3 + n_4) - n_3 n_4 (n_1 + n_2)\} \quad (3.7)$$

where, $|\mathbf{k}_1| \leq |\mathbf{k}_2|$ (or $\omega_1 \leq \omega_2$) is assumed without loss of generality from the symmetry of Eq. (3.5). The variables are non-dimensionalised by

$$\tilde{\theta}_1 = \theta_1 - \theta_a, \quad \tilde{\theta}_2 = \theta_2 - \theta_a, \quad \tilde{\theta}_3 = \theta_3 - \theta_4, \quad \tilde{\omega}_1 = \omega_1 / \omega_4, \quad \tilde{\omega}_2 = \omega_2 / \omega_4, \quad \tilde{\omega}_3 = \omega_3 / \omega_4,$$

$$\text{and } \tilde{\Omega} = \ln \tilde{\omega}_3.$$

The denominator S arising from $\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)$ is given by

$$S = \left| 1 + (\omega_1 / \omega_2) \{ (\omega_1 / \omega_2)^2 - (|\mathbf{k}_a| / \omega_2^2) \cos(\theta_1 - \theta_a) \} \right| \quad (3.8)$$

As Masuda noted, a numerical instability in the integration of Eq. (3.7) is caused mainly by inappropriate treatment of singular points. Masuda hence solved this instability problem by analytically deriving an approximate solution of Eq. (3.7) around the singular points.

In order to improve the performance of wave models with the above technique of Masuda and to gain better physical understanding of the spectral evolution, Komatsu and Masuda (1996) developed a new scheme called the RIAM method (RIAM = Research Institute for Applied Mechanics, Kyushu University, Japan) for calculating the non-linear energy transfer on the basis of the rigorous method of Masuda (1980). This new scheme was developed by taking advantage of the symmetry of the integrand as in Hasselmann and Hasselmann (1981) or Resio and Perrie (1991), and by truncating less significant configurations of resonance to achieve shorter computational time without loss of accuracy.

As Komatsu and Masuda (1996) mentioned, there are two kinds of symmetries in the resonant interaction. The first is based on the well-known nature of non-linear resonant interactions among gravity waves expressed by Eq. (3.1). As explained in Hasselmann and Hasselmann (1981), $\delta n(\mathbf{k}_i) d\mathbf{k}_i / \delta t$ ($i = 1, 2, 3, 4$) have the following relationship:

$$\frac{\delta n(\mathbf{k}_1)}{\delta t} d\mathbf{k}_1 = \frac{\delta n(\mathbf{k}_2)}{\delta t} d\mathbf{k}_2 = -\frac{\delta n(\mathbf{k}_3)}{\delta t} d\mathbf{k}_3 = -\frac{\delta n(\mathbf{k}_4)}{\delta t} d\mathbf{k}_4 \quad (3.9)$$

where $\delta n(\mathbf{k}) / \delta t$ indicates the action transfer that is due to this particular resonance combination. As shown in Equation (3.6), $\delta n(\mathbf{k}_i) d\mathbf{k}_i / \delta t$ ($i = 1, 2, 3, 4$) are of equal magnitude but are different in sign.

Accordingly, if we calculate $\delta n(\mathbf{k})/\delta t$ for one component of the resonant four waves, then we immediately know $\delta n(\mathbf{k})/\delta t$ for the other three components. The other type of symmetry is associated with the geometrical similarity of resonance configurations. One is the mirror image of a resonance combination that has the same interaction coefficient as the original one. The other is a rotation of a resonance combination that also gives the same interaction coefficient.

Now, we specify a particular wave number vector \mathbf{k}_4 (with ω_4 and θ_4) at which the non-linear energy transfer is to be evaluated, and then assume the sequence of frequencies as follows, considering the first kind of symmetry of the non-linear wave-wave interaction so as to eliminate the overlap computations.

$$\omega_3 \leq \omega_1 \leq \omega_2 \leq \omega_4 \quad (3.10)$$

For the computation of realistic continuous energy transfer of $\partial n(\omega, \theta)/\partial t$, the computation must be carried out with the loops of frequency ω_4 and direction θ_4 . The computation of the configuration of resonant interactions are to be performed in advance with the computation of variables such as \tilde{G} , S , \dots in the Boltzmann integral for both regular and singular points. The details of the computation procedure are explained in Masuda (1980) and Komatsu and Masuda (1996).

The SRIAM method (approximate method)

The RIAM method turned out to have the same degree of accuracy as Masuda's rigorous method. Although the RIAM method is 300 times faster than Masuda's method, it is still a few thousands times slower than the DIA, simply because the RIAM method processes thousands of resonance configurations.

Hence, Komatsu (1996) developed a new scheme of practical efficiency with a slightly lower level of accuracy than the RIAM method. The method is called the Simplified RIAM (SRIAM) method, which processes 20 representative configurations chosen by some optimisation. The SRIAM method can be expressed by the following equation.

$$T(\omega_4, \theta_4) = \frac{\partial \phi_4}{\partial t} = (2\omega_4^{23}) \sum_{i=1}^{20} C_i \tilde{K}_i(\tilde{\theta}_1, \tilde{\Omega}, \tilde{\theta}_3) \{n_1 n_2 (n_3 + n_4) - n_3 n_4 (n_1 + n_2)\}_i \quad (3.11)$$

where $\tilde{K}_i(\tilde{\theta}_1, \tilde{\Omega}, \tilde{\theta}_3) = 8\tilde{\omega}_1^3 \omega_3^4 \tilde{G} S^{-1} \Delta \tilde{\theta}_3 \Delta \tilde{\Omega} \Delta \tilde{\theta}_1$, and $C_i, (i=1, \dots, 20)$ are the coefficients.

Komatsu (1996) listed the optimum 20 combinations of the resonant configurations as well as the optimum positive coefficients C_i tuned by some optimisation, where 7 configurations are chosen for singular points and the other 13 configurations are chosen for regular points from $(\tilde{\theta}_1, \tilde{\Omega}, \tilde{\theta}_3)$ space.

It is noted that in the method of Komatsu (1996) the optimum resonant configurations and the coefficients C_i depend on how many configurations we choose as well as how many directional and frequency bins

constitute the directional spectrum in the model. That is, the optimum resonant configurations and the coefficients C_i have to be re-determined for each different computation conditions. For these reasons, a simpler method is preferable to determine the optimum configurations and the coefficients C_i .

4. Setup of inter-comparison study

The first step of the inter-comparison study was to generate a set of discrete (deep water) 2D wave spectra and distribute them among the participants. Each 2D spectrum is written to file in ASCII format and includes the discrete frequencies, discrete directions and the 2D variance density as a function of frequency and direction in the form of a 2D matrix. The first spectrum is considered the base case and all other spectra are variations to this spectrum. The characteristics of the base spectrum are as follows:

- Number of frequencies 50, logarithmically distributed from 0.03 Hz to 3 Hz, yielding a frequency resolution of 10%;
- Number of directions 36, equally distributed over the full circle, $\Delta\theta=10^\circ$, starting at $\theta=0^\circ$;
- JONSWAP frequency spectrum with $\alpha=0.0175$, $f_p = 0.2$ Hz, peak enhancement factor $\gamma=3.3$, shape factors of peak $\sigma_a=0.07$, $\sigma_b=0.09$, and a parametric tail with power -5. In the discrete frequency spectrum the peak occurs at bin 19 with $f=0.1965$ Hz;
- Directional distribution according to $D(\theta)=\cos^{2s}(\theta/2)$, with $s=2$.

Following the considerations presented in Section 2, a first set of test spectra has been generated. As the inter-comparison progresses, additional spectra are likely to be added. The characteristics of all test spectra are summarized in Table 1.

#test	Description
1	Base case
2	Lower upper frequency, $f_{max}=1$ Hz, number of frequencies, $nfr=37$
3	Higher upper frequency, $f_{max}=5$, number of frequencies, $nfr=54$
4	Smaller number of frequencies: $nfr=30$, resolution 17% (i.e. relative difference between subsequent frequencies, an increase in resolution is equivalent with a decrease of this difference)
5	Higher number of frequencies, $nfr=80$, resolution 6%
6	Smaller number of directions: $ndir=24$
7	Higher number of directions: $ndir=72$
8	Higher Phillips constant: $\alpha=0.035$
9	Higher peak frequency: $fp=0.3$ Hz
10	Smaller peak enhancement factor: $\gamma=1$ (effectively a Pierson-Moskowitz spectrum)
11	Higher peak enhancement factor: $\gamma=7$
12	Power of spectral tail -4 (See Holthuijsen 2007, for the correct modification to JONSWAP spectrum)
13	Narrower directional spreading, $s=10$;
14	Directional distribution shifted over half a bin, $\Delta\theta=5^\circ$;
15	Directional distribution shifted over a full bin, $\Delta\theta=10^\circ$;
16	Energy density halved at twice the peak frequency for all directions. In the discrete frequency spectrum this halving occurs at the 29 th frequency bin with $f=0.4168$ Hz.

Table 1: Summary of test cases used in the inter-comparison study, status October 2013.

The set of discrete spectra is stored in separate data files with clear headers that can be processed by each participating member. It is recommended that each computational method is setup in such a way that automatic processing becomes feasible. The computed non-linear 2D transfer rates and integrated 1D-transfer rates ($S_{nl4}(f)$ or $S_{nl4}(\theta)$) should then be collected as ASCII data files in a similar data format to enable easy processing for the inter-comparison.

In the following section a selected number of results of the inter-comparison study is shown, highlighting some differences and yielding some questions needing further attention.

5. Preliminary results

The first step was to inter-compare the frequency and direction dependent transfer rates $S_{nl4}(f)$ and $S_{nl4}(\theta)$ for the base case. The result is shown in Figure 1, in which the RIAM result is multiplied with a factor 2 to agree with the WRT result (similarly the WRT result could be divided by a factor 2). It is noted that this factor was also mentioned by Tanaka (2007). The source of this factor is possibly related to (not) accounting for a symmetry in either computational method. The agreement between the frequency dependent transfer rates is reasonable although some differences occur in the magnitude of the first negative lobe and in the level at the higher frequencies. The differences at the positive and negative peak are probably related to minor differences in (interpolated) spectral density which are magnified in the computed transfer rate due to the (highly) non-linear cubic behavior of the kinetic integral. The direction dependent transfer rate $S_{nl4}(\theta)$ is shown in the right panel of Figure 1. Here, a systematic difference occurs between both methods where the RIAM has smaller positive lobes, and a larger negative lobe. This systematic difference might lead to different directional properties of evolved wave spectra in dynamic tests.

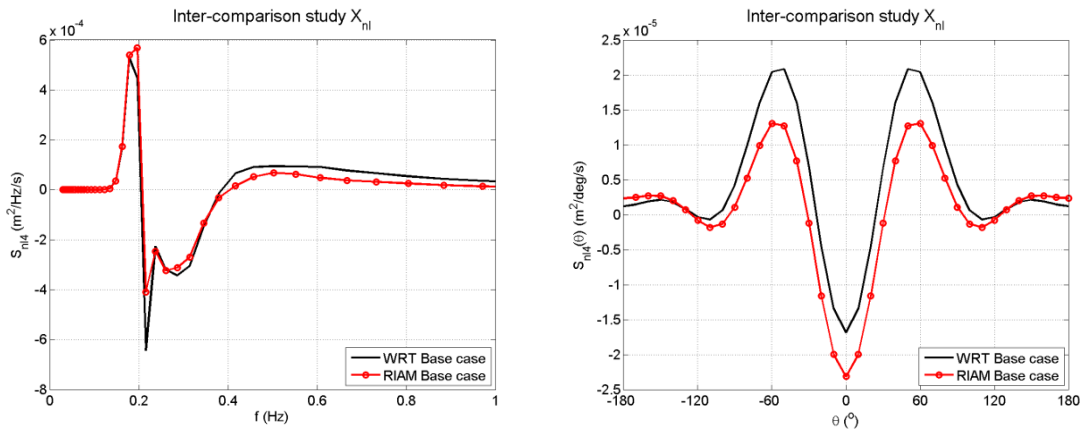


Figure 1: Inter-comparison of the non-linear transfer rate computed with the WRT and RIAM method for the base case. Frequency dependent transfer rates (left panel), direction dependent transfer rates (right panel). The RIAM values are multiplied with a factor 2.

The effect of varying the number of frequencies is shown in Figure 2 for both the WRT (left panel) and RIAM method (right panel). Apart from the differences noted in the comparison of the base case (Figure 1), it follows that both methods show similar results when the frequency resolution is changed. The most notable result is that in the coarse resolution the local peak at $f=0.23$ Hz disappears.

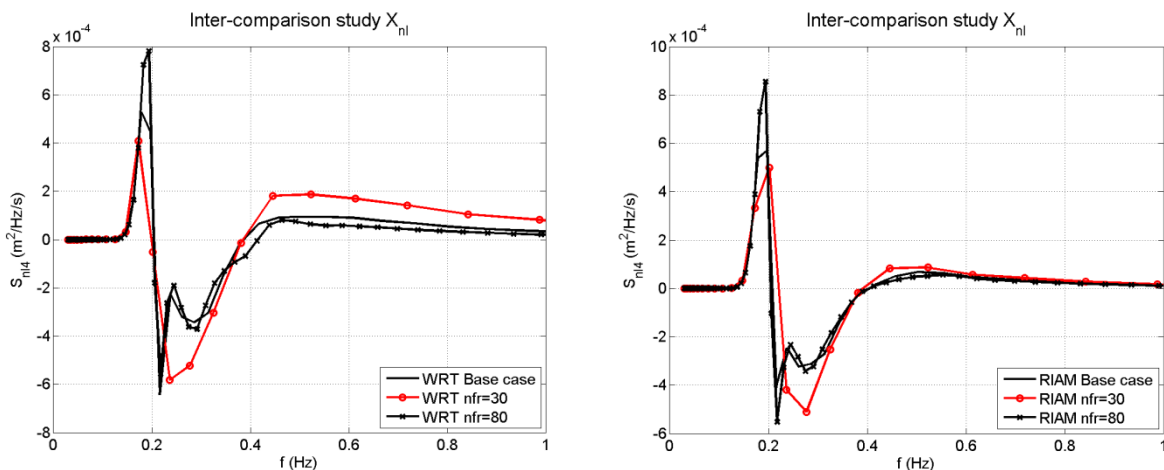


Figure 2: Inter-comparison of the non-linear transfer rate computed with the WRT (left panel) and RIAM method (right panel) for the cases 1, 4 and 5 showing the effect of frequency resolution.

The effect of varying the directional resolution on the directional transfer rate is shown in Figure 3 for both the WRT and RIAM method. On the one hand the WRT is a bit more sensitive to variations in directional resolution than the RIAM method at the positive lobes. On the other hand the RIAM method has a larger negative peak at $\theta=0^\circ$.

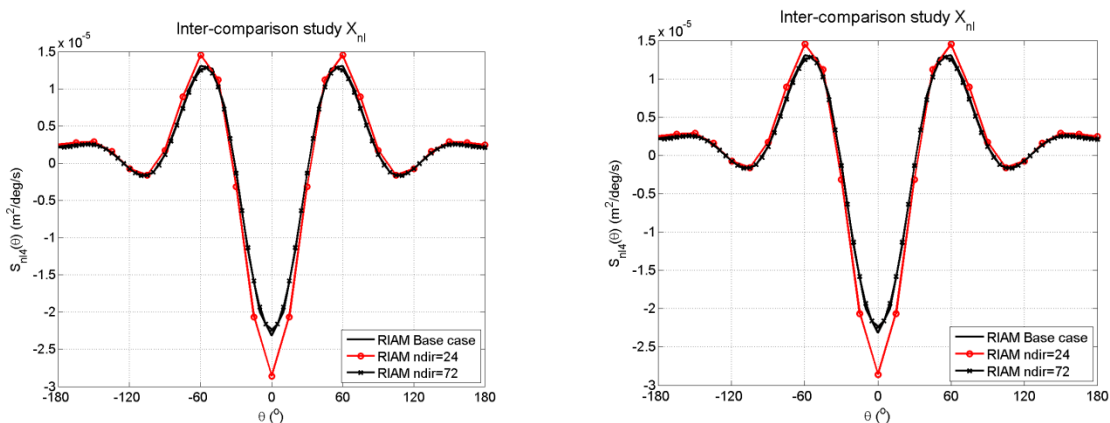


Figure 3: Inter-comparison of the non-linear transfer rate computed with the WRT (left panel) and RIAM method (right panel) for the cases 1, 4 and 5 showing the effect of directional resolution.

The effect of narrowing the directional spreading on the non-linear transfer rate is shown in Figures 4 and 5 for both the WRT method (left panels) and RIAM method (right panels). Figure 4 shows the frequency dependent transfer rates for the base case and case 13 and Figure 5 shows the related results for the directionally dependent transfer rates. A first look at the results shows that both methods show a similar response to a change in the directional spreading of the input spectra. Narrowing the directional spreading yields a much more peaked transfer rate in both frequency and direction space.

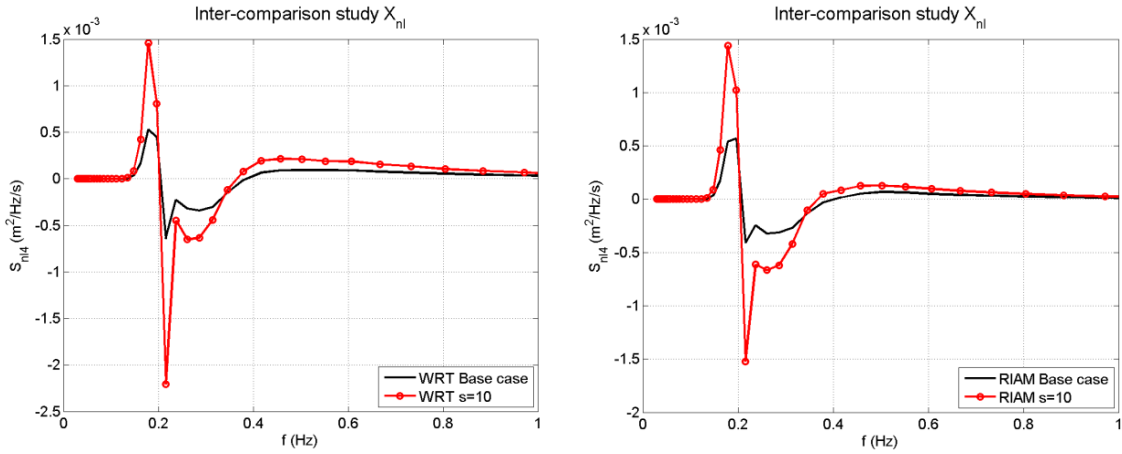


Figure 4: Inter-comparison of the frequency dependent non-linear transfer rate computed with the WRT (left panel) and RIAM method (right panel) for the cases 1 and 13 showing the effect of the directional spreading.

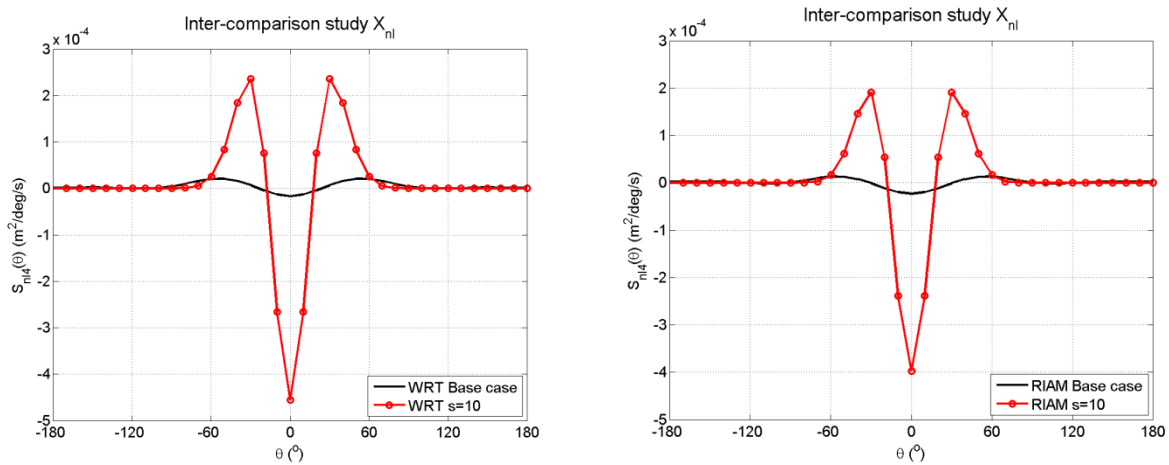


Figure 5: Inter-comparison of the directionally dependent non-linear transfer rate computed with the WRT (left panel) and RIAM method (right panel) for the cases 1 and 13 showing the effect of the directional spreading.

A critical test for all methods is the response to a perturbation in spectral density. Resio and Perrie (1991) and Young and Van Vledder (1993) describe tests with a spectrum in which the spectral density is halved at twice the peak frequency for all directional bins. The resulting effect on the non-linear transfer rate for the WRT, RIAM and SRIAM method is shown in Figure 6. All methods show a strong response to this

perturbation. The results show that the WRT and RIAM method show a similar response to the perturbation whereas the SRIAM shows some systematic differences. As discussed before, small differences in the peak magnitudes are probably due to small differences in (interpolated) spectral densities.

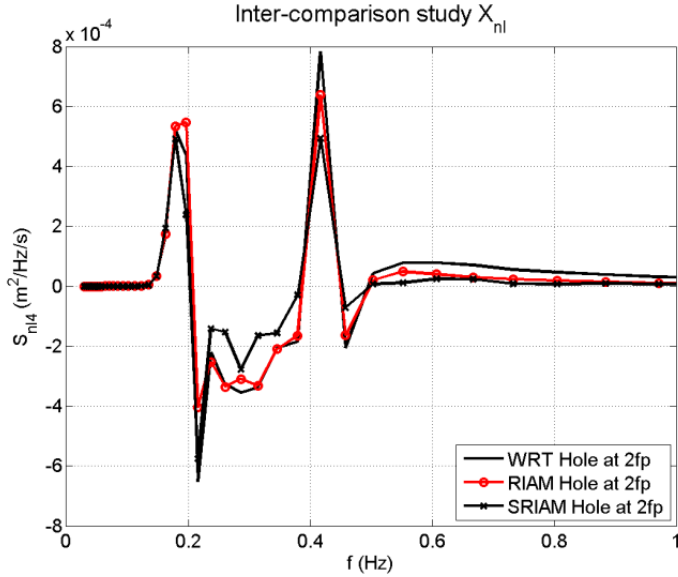


Figure 6: Inter-comparison of the frequency dependent non-linear transfer rate computed with the WRT, RIAM and SRIAM method showing the effect of a gap in the spectrum at $f=2f_p$ (cases 1 and 19).

So far, only one test result was shown including an approximate method (Figure 6). As shown by Van Vledder et al. (2000) significantly increasing the frequency resolution (from 10% to 3%) may lead to unexpected results in dynamic tests. It is therefore of interest to apply the frequency resolution tests also to the DIA and the SRIAM methods. The results are presented in Figure 7. (It is noted that both authors have their own version of the DIA method, which provide equal results.)

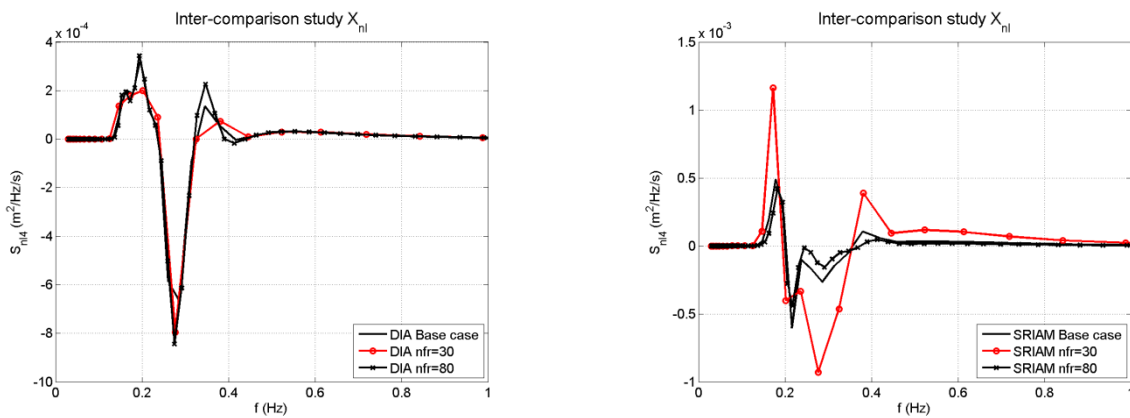


Figure 7: Inter-comparison of the frequency dependent non-linear transfer rate computed with the DIA (left panel) and SRIAM (right panel) showing the effect of frequency resolution.

The results for the DIA show that increasing the frequency resolution leads to more peaked positive lobes. For the SRIAM, an increase in frequency resolution has little effect, but decreasing the frequency resolution leads to significantly larger peak magnitudes and also a shift in the position of the negative lobe. Whether this behaviour also occurs in dynamic tests cannot yet be said.

6. Discussion

The results shown in this paper demonstrate the value of the inter-comparison. An important finding is the difference in magnitude of the WRT and RIAM method. As the shape of the resulting non-linear transfer rates are very similar, we expect that the source of this constant difference can easily be found. The comparison also shows minor differences in peak values. Although the input is equal for all methods, differences in e.g. interpolations lead to small differences in action density, which are strongly magnified by the highly non-linear cubic nature of the kinetic integral.

Static tests might reveal small differences in results. On the one hand these differences might point to certain weak points in the computational method. On the other hand these differences might become irrelevant in dynamic tests as there the overall performance of the host wave model is more important than the accuracy of the underlying source terms.

A frequently asked question about the Discrete Interaction Approximation (DIA) is why the frequency resolution should be about 10%. So far, there is no answer available that is based on first principles. Instead, this choice can be considered as the effect of tuning. When the DIA was introduced by Hasselmann et al. (1985) the 10% resolution was presented as an optimal choice in relation to the choice of the λ - factor relating the frequencies of interacting wave number vectors, such that growth curves from literature could be reproduced. Changing the frequency resolution would then imply also a change of the λ - factor, or vice-versa, to obtain optimal results. Adding more generally shaped wave number configurations will probably reduce the sensitivity of the non-linear term to frequency resolution. This hypothesis might be tested against the GMD of Tolman (2013).

This paper is the first step towards an objective methodology for judging the performance of computational methods for computing the non-linear four-wave interactions in discrete spectral wave models. This work was inspired by the need to obtain confidence in solving the complicated Boltzmann integral, but also to provide a framework to judge or challenge (sometimes incompletely substantiated) claims in literature about the efficiency of newly proposed methods for computing the non-linear transfer rate.

The proposed framework can also be used to investigate of effect of internal settings of each computational method. Examples are the method to obtain the spectral density at the wave number vectors involved in a quadruplet or the quadrature method involved in solving internal integrals. A good understanding of the ‘workings’ of each computational methods is therefore important.

Further tests are needed to determine whether the methods actually solve the same equation. Consistency tests in which the spectral resolution is increased is one way of achieving this goal. Another way is to

compare results with those from methods like the EXACT-NL model (Hasselmann and Hasselmann, 1981), or with analytic results, like the narrow peak approximation described by Dungey and Hui (1979).

In addition, further work is needed to develop the inter-comparison methodology into an accepted tool in wind-wave modelling. The authors are open to further suggestions about static and dynamic tests. Regarding the statics tests, additional tests will probably pop-up as experience is gained with these tests. Regarding the dynamics tests inspiration can be obtained from the method applied by Tolman and Grumbine (2013) in finding optimal coefficients of the Generalized Multiple DIA (GMD) of Tolman (2013).

The present paper is still limited as far as it concerns all possible tests and methodologies. Additional tests will be reported elsewhere and the inter-comparison study also needs to be extended with other exact and approximate methods available in literature.

7. Conclusions and recommendations

An objective method is proposed to objectively inter-compare methods for the computation of non-linear four-wave interactions in discrete spectral wave models. The tests are useful to identify differences in results and to find the origin of these differences. The first results show systematic differences and minor differences in the resulting non-linear transfer rates. The origin of these differences need to be determined, e.g. by performing a consistency test in which the spectral resolution in both frequency and direction space is gradually increased. It is also recommended to compare results of the computational methods with other results from literature or with results of narrow peak approximations for which analytic solutions are available.

The set of test spectra can be obtained from the first author.

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