Nonlinear dynamics of shoaling gravity waves

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INTRODUCTION

The nearshore propagation and transformation of wind-driven ocean waves is affected by medium variations (refraction), nonlinear wave-wave interactions, and dissipation. Understanding how these processes affect wave statistics, and thus wave-driven dynamics, is important for coastal dynamics and nearshore transport processes. In the present paper we will focus on the role of triad nonlinearity and dissipation on the statistics of weakly dispersive waves in the very nearshore, from just outside the surf zone to the beach, also known as the wave shoaling and surf zone.

Because of the lack of frequency dispersion in shallow water, an asymptotic closure such as the resonant interaction closure approximation (Hasselmann 1962), which has been central to the development of operational wave models (see e.g. Tolman 1991; Komen et al. 1994), appears not possible in the nearshore. Numerous statistical models for shallow-water wave propagation have been developed, either based on the so-called Zakharov kinetic integral (Eldeberky et al. 1996), Boussinesq-type amplitude equations (e.g. Herbers and Burton 1997) or amplitude evolution equations including full dispersion in the linear terms and the coupling coefficient (Agnon and Sheremet 1997; Eldeberky and Madsen 1999). Invariably these models apply a so-called quasi-normal closure, a semantics borrowed from turbulence literature, with either full discard of the fourth cumulant or a heuristic approximation (see e.g. Rasmussen 1998 for a review). Generally, these stochastic models either solve a coupled set of equations for the spectrum and bi-spectrum (Herbers and Burton 1997; Eldeberky and Madsen 1999) or explicitly integrate the bispectral evolution equation – at the expense of additional assumptions – to obtain a single transport equation for the energy spectrum (e.g. Eldeberky et al. 1996).

The Quasi-Normal (QN) approximation is shown to be accurate up to Ursell numbers around 1.5 (e.g. Agnon and Sheremet 1997; Norheim et al. 1998). However, in regions of stronger nonlinearity (and possibly strong dissipation), this approximation becomes inherently inaccurate, to the extent that predictions can exaggerate non-Gaussian features in the wave field, and can include negative ‘energies’, (see e.g. Ogura 1962). Herbers et al. (2003) proposed a closure modification by adding a dissipation-controlled relaxation term to the bi-spectral evolution equation, which improves the

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model-predicted nonlinear dynamics in regions of strong nonlinearity. In the present work we extend this closure modification to include a dependency on nonlinearity and analyze its effects on the nonlinear dynamics.

The objective in the present work is twofold. First we discuss the stochastic modeling framework, define the Quasi-Normal closure approximation, introduce a new closure approximation by adding a relaxation term to replace the fourth-cumulant contribution to the bispectrum equation, and illustrate the differences in the closure characteristics through comparison of the stochastic model to Monte Carlo simulations. In the second part we implement a dissipation model, and compare the resulting stochastic model (including relaxation term) to a set of laboratory observations of waves over a beach.

A NON-ASYMPTOTIC, ONE-POINT CLOSURE APPROXIMATION

We consider the propagation of waves on the surface of an inviscid and incompressible fluid, and adopt a conventional Cartesian description with the origin of the reference frame at the undisturbed free surface of the fluid. We let z denote the vertical, positive pointing upward, and $\boldsymbol{x} = (x, y)$ the horizontal dimensions. The waves are assumed to propagate into the half plane of $x \in \mathcal{R}^+$ where we will refer to x as the principal direction, and y as the lateral direction. The surface elevation $\zeta(\boldsymbol{x}, t)$ associated with the random (but stationary) wave field is represented by a Fourier sum over frequency and (angular) directional components

$$\zeta(\boldsymbol{x}, t) = \sum_{p_1, q_1 = -\infty}^{\infty} \frac{A_1^1(\boldsymbol{x})}{\sqrt{V_1^1}} \exp\left[i(\lambda_1 y - \omega_1 t)\right]. \tag{1}$$

Here $\omega_1 = \omega_{p_1} = p_1 \Delta\omega$, $\lambda_1 = \lambda_{q_1} = q_1 \Delta\lambda$ with $\Delta\omega$ and $\Delta\lambda$ the discrete angular frequency spacing; the numerical sub- and superscript on the wave variable A_1^1 is short for $A_{\omega_1}^{\lambda_1}$; the V_1^1 is the principal component of the group speed vector, written as

$$V_1^1 = \frac{d\omega_1}{d\varkappa_1^1} = \frac{\varkappa_1^1}{k_1} \frac{d\omega_1}{dk_1}. \tag{2}$$

Here k_1 is the wavenumber related to ω_1 through the linear dispersion relation, and the principal wavenumber component $\varkappa_1^1 = \mathrm{sgn}\,(\omega_1)\sqrt{k_1^2 - \lambda_1^2}$; the sgn function is included to ensure that the surface elevation associated with the forward-propagating wave field is real. Since we consider the wave field statistically homogenous and stationary, the numerical sub- and superscripts for frequency and lateral wavenumbers respectively are identical everywhere so that, for readability, we will drop the superscripts from our notation in the following. Note that the Fourier amplitudes A_1, as defined in (1), can be thought of as flux amplitudes, such that in a conservative wave field $|A_1|$ is constant.

The angular-spectrum decomposition implies a forward-scattering approximation (see e.g. Janssen et al. 2008), which is a convenient (and realistic) framework for our discussion of the dynamics of nearshore wind waves incident onto a dissipative beach. A more generic, isotropic description of the evolution of wave correlators in arbitrary depth is presented elsewhere (Smit and Janssen 2011).

From the nonlinear boundary value problem for surface gravity waves, and in the absence of ambient currents, the wave field evolution over distances of $O(\epsilon^{-1})$ can be described by (see e.g. Janssen et al. 2006; Janssen et al. 2008)

$$\left[\frac{d}{dx} - i\varkappa_1 + \nu_1\right] A_1 = i \sum_{\omega_2, \lambda_2} \mathcal{W}_{(1-2),2} A_{(1-2)} A_2 \tag{3}$$

where $\mathcal{W}_{12}$ is the second-order interaction coefficient (see e.g. Janssen et al. 2006 for details), and ν_1 is a dissipation term to account for depth-induced wave breaking; the details of the latter are given later, where we discuss the complete model implementation.

Although no explicit assumptions were introduced regarding the dispersivity of the waves, the evolution equation (3) is a shallow-water model since the lack of a cubic nonlinear term limits its use to areas where quadratic interactions approach resonance and leading-order energy transfers are on relatively short length scales (say $O(10)$ wavelengths).

To estimate the statistics associated with the deterministic model (3), we can develop evolution equations for the statistical moments (or cumulants) through ensemble averaging (e.g. Janssen et al. 2008). Through the presence of the nonlinear term the hierarchy of equations for the statistical moments is open (see e.g. Orszag 1970; Holloway and Hendershott 1977; Lesieur 1997; Salmon 1998 and many others) and some form of truncation, or *closure*, is needed. Here we consider a closure by approximating the fourth cumulant in terms of lower-order statistical moments (or discarding it). The statistical model thus consists of coupled equations for the energy spectrum and bispectrum, which can be written as

$$\left[\frac{d}{dx} + 2\nu_1\right]\mathcal{E}_1 = -2\iint d\omega_2 d\lambda_2 \mathcal{W}_{(1-2),2}\Im\{\mathcal{C}_{23}\} \tag{4a}$$

$$\left[\frac{d}{dx} - i\Lambda_{12} + \nu_{12}\right]\mathcal{C}_{12} = 2i\mathcal{Q}_{12} + \mathcal{K}_{12}^{(4)} \tag{4b}$$

where

$$\mathcal{E}_1 = \lim_{\Delta\omega,\Delta\lambda\to 0}\frac{\langle|A_1|^2\rangle}{\Delta\omega\Delta\lambda}, \qquad \mathcal{C}_{12} = \lim_{\Delta\omega,\Delta\lambda\to 0}\frac{\langle A_1 A_2 (A_{(1+2)})^*\rangle}{\Delta\omega^2\Delta\lambda^2}. \tag{5}$$

and where we defined $\Lambda_{12} = \varkappa_1 + \varkappa_2 - \varkappa_{(1+2)}$, $\nu_{12} = \nu_1 + \nu_2 + \nu_{(1+2)}$, and $\mathcal{Q}_{12} = \mathcal{E}_{(1+2)}\left(\mathcal{W}_{(1+2)(-2)}\mathcal{E}_2 + \mathcal{W}_{(1+2)(-1)}\mathcal{E}_1\right) - \mathcal{W}_{12}\mathcal{E}_1\mathcal{E}_2$. The $\mathcal{K}_{12}^{(4)}$ (on the right of (4b)) represents the fourth-cumulant contributions to the bispectrum evolution, which will be discussed later.

By not directly restricting the third cumulant in equation (4), the model retains its principal nonlinearity, but without introducing the increased complexity (and dimensionality) of higher-order cumulant evolution equations. This seems a sensible choice, in particular since it is not clear that adding higher cumulants would actually improve the statistical model. After all, our deterministic model includes only quadratic nonlinearity, whereas the evolution of cumulants higher than the third are affected also by higher-order nonlinearities; thus, although adding cumulants beyond the third may make the stochastic model a more complete representation of the statistics implied by the underlying (incomplete) deterministic model, it would not necessarily improve the representation of nonlinear statistics of natural wave fields. Moreover, we consider nearshore wave propagation onto a dissipative beach, where the dynamics are strongly affected by dissipation (dominated by depth-induced breaking), the details of which are still poorly understood (see e.g. Kaihatu et al. 2007). In other words, given the limitations of the deterministic model and our limited understanding of the effect of other physical processes on the closure dynamics (in particular dissipation), the added complexity of evolving higher-order cumulants would not necessarily improve the model representation of nonlinear wave statistics.

Quasi-Normal (QN) approximation

A consistent Gaussian closure would imply that all cumulants beyond the second vanish, so that the resulting wave statistics are indeed Gaussian. However, this would thus result in a linear model in which the spectrum components decay solely due to the presence of dissipation. In other words, such a closure would eradicate the system of nonlinearity completely. Alternatively, to retain at least the principal nonlinearity in the system, the Quasi-Normal (QN) closure approximation assumes that all cumulants higher than the third are zero, which implies that $\mathcal{K}_{12}^{(4)} = 0$, but without directly constraining the third-cumulant (or bispectrum).

In our case, setting $\mathcal{K}_{12}^{(4)} = 0$ and combining the set (4) (by integrating (4b) and substituting into (4a)) results in

$$\left[\frac{d}{dx} + 2\nu_1\right]\mathcal{E}_1 = -4\sum_{\omega_2,\lambda_2} \mathcal{W}_{(1-2)2}\Re\{\int_0^x \mathcal{H}^{\text{QN}}_{(1-2)2}(x,x')\mathcal{Q}_{(1-2)2}(x')\,dx'\} \tag{6}$$

where

$$\mathcal{H}_{12}^{\text{QN}}(x,x') = \exp\left[\int_{x'}^{x} \left(i\Lambda_{12}(s) - \nu_{12}(s)\right)\,ds\right] \tag{7}$$

The integral on the right of equation (6) can be thought of as a memory integral, where $\mathcal{H}$ takes on the role of an influence function, which determines how much, and in which way, the past states of the fourth moment affect the future state of the system[1]. In the limit of weak dispersion (where $\Lambda_{12} \to 0$), the memory of the system in the QN approximation is principally constraint only by dissipation.

A principal (and known) shortcoming of the QN approximation is the lack of de-correlation of the three-wave correlations as would be associated with the background nonlinear random wave field. After all, by discarding the fourth-cumulant, the evolution of the three-wave correlations is effectively decoupled from higher-order correlations in the random wave field. In other words, from the viewpoint of the stochastic model, the evolution of the spectrum and bispectrum depend only on dispersion, dissipation and their mutual coupling (nonlinearity), but not on the presence of higher-order correlations. In practice this will result in three-wave correlations that are too strong (by lack of nonlinear de-correlation, see e.g. Orszag 1970; Holloway and Hendershott 1977), and initial tendencies in the system that are retained too long. In areas of strong nonlinearity these characteristics can result in unrealistic physics, including energies becoming negative in the energetic ranges of the spectrum.

Quasi-Normal Relaxation (QNR) approximation

To improve the closure characteristics of the QN approximation in the context of a one-point closure approximation, we model the fourth-cumulant contribution as a linear damping term of the form (see e.g Holloway and Hendershott 1977; Herbers et al. 2003)

$$\mathcal{K}_{12}^{(4)} = -\mu_{12}\mathcal{C}_{12}, \tag{8}$$

[1]Although our model equations evolve (stationary) statistics through space, we will refer to past and future to indicate regions of the domain where the wave statistics are determined (known) and where it is yet to be determined.

where the μ_{12} can be thought of as a relaxation coefficient, which acts as a damping term on the three-wave correlations and thus allows the system to return to a near-Gaussian state in the presence of strong nonlinearity. Substituting (8) in equation (4b) results in the influence function (7) being redefined as

$$\mathcal{H}_{12}^{\mathrm{QNR}}(x,x') = \exp\left[\int_{x'}^{x} \left(i\Lambda_{12}(s) - \nu_{12}(s) - \mu_{12}(s)\right)\, ds\right] \tag{9}$$

In our one-point closure approximation (we do not explicitly consider spatial cross-correlations of the wave field), there appears no possibility to estimate μ_{12} from first principles. Therefore, to close the model we simplify μ_{12} by treating it as a (real and positive) damping term that depends on the strength of the changes in energy across the triad due to nonlinearity and dissipation, so that

$$\mu_{12}(x) = \beta\frac{|\mathcal{DS}_1 + \mathcal{DS}_2 + \mathcal{DS}_{(1+2)}| + |\mathcal{NL}_1 + \mathcal{NL}_2 + \mathcal{NL}_{(1+2)}|}{\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_{(1+2)}} \tag{10}$$

where β is a tunable constant (anticipated to be $O(1)$) and

$$\mathcal{NL}_i = -4\sum_{\omega_2,\lambda_2} \mathcal{W}_{(1-2)2}\Re\{\int_0^x \mathcal{H}_{(1-2)2}^{\mathrm{QNR}}(x,x')\mathcal{Q}_{(1-2)2}(x')\,dx'\} \tag{11}$$

$$\mathcal{DS}_i = 2\nu_i\mathcal{E}_i. \tag{12}$$

The relaxation term proposed here provides a simple (and quasi-empirical) means to recover some of the principal effects of the presence of the random wave background on the evolution of the three-wave correlations (bispectrum). Whereas it does not capture the effects of the fourth-cumulant in detail, it provides a de-correlation mechanism that mimics the damping effects associated with the built-up of higher-order correlations, which is important in regions of strong nonlinearity.

Closure characteristics

To illustrate the behavior of the QN and QNR model, in the absence of dissipation, we compare Monte-Carlo simulations (using the deterministic model (3)) to the evolution predicted by the QN model (eqs (6) with (7)) and QNR model (eqs (6) with (9)). The objective here is to illustrate the differences in these closure approximations, and to identify the principal effects of the relaxation term on the evolution of the statistics. In our simulations, the models are initiated at $x = 0$ with a (double-sided) frequency spectrum of the form

$$\mathcal{E}(\omega, x=0) = V(\omega)\frac{H_{\mathrm{rms}}^2}{16\sigma\sqrt{2\pi}}\exp\left[-\frac{(\omega-\omega_p)^2}{2\sigma^2}\right] \tag{13}$$

where H_{rms} is the root-mean-square wave height, $V(\omega) = \partial_k\omega$, and $\sigma = 0.06\,\mathrm{rad/s}$. The spectrum is discretized in 64 equidistant frequencies with $\Delta\omega = 0.05\pi\,\mathrm{rad/s}$, and the integration is performed using a fixed-step-size Runge-Kutta scheme ($\Delta x = 0.25\,\mathrm{m}$). The peak frequency of the wave field is $\omega_p = 0.2\pi\,\mathrm{rad/s}$. Comparison of spectra after 10 wavelengths ($x/L_0 = 10$) and 25 wavelengths ($x/L_0 = 25$) shows that the spectra from the Monte-Carlo simulations are generally smoother than the more spiky spectra from the QN model (figure 1). Further, the QN model spectra become negative at several

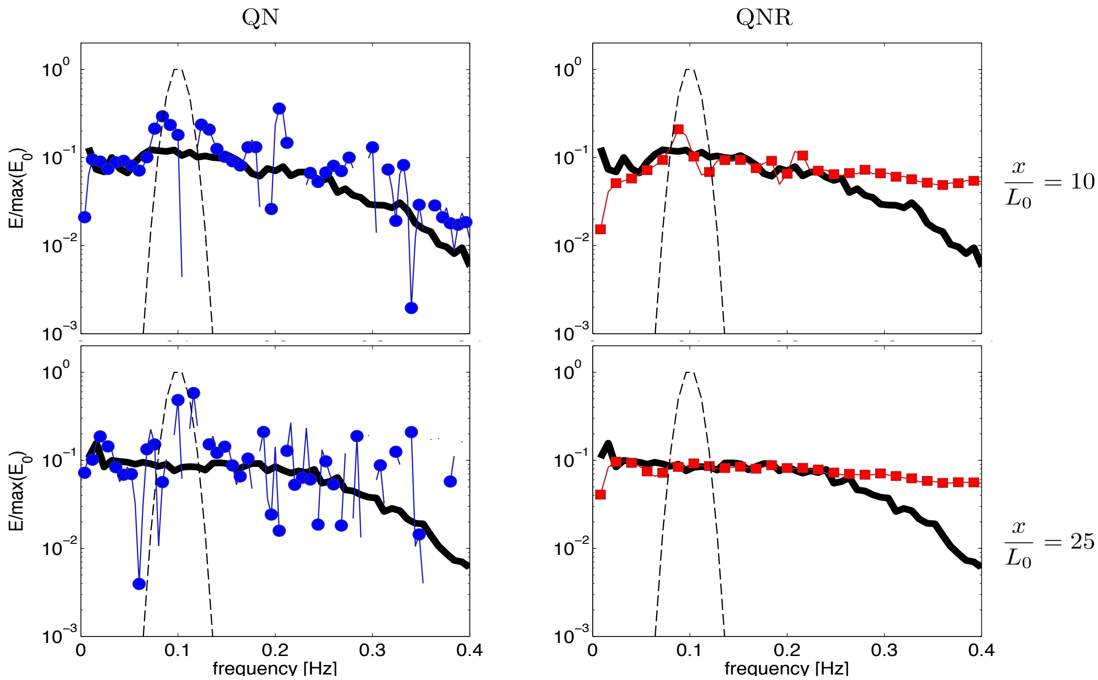

FIG. 1. Comparison frequency spectra from Monte-Carlo simulations (black solid line), QN model (left panels, blue circle markers), and QNR model (right panel, red square markers) at $x/L_0 = 10$ (top panels) and $x/L_0 = 25$ (bottom panels). The initial spectrum at $x = 0$ is indicated by the dashed line.

places in the frequency range (including the energy-carrying range), which are shown as gaps because of the logarithmic scale. The QNR-simulated spectra are generally in much better agreement with the Monte-Carlo simulations, in particular in the energy-carrying range of the spectrum. However, the added relaxation in the QNR model does result in some overestimation of energy levels in the spectral tail (figure 1).

Energy transfers near the (initial) peak ($\omega = \omega_p$) and its first harmonic ($\omega = 2\omega_p$) are strongly exaggerated by the QN model (figure 2). The QNR closure approximation results in much more realistic energy transfers by allowing the damping of three-wave correlations. In the absence of dissipation, and without nonlinear-induced relaxation, the QN approximation cannot effectively reduce the strength of three-wave correlations present in the wave system; this results in the continuous back-and-forth transfer of energy across triads, which is not seen in the Monte-Carlo simulation (see figure 2).

A STATISTICAL MODEL FOR SHALLOW-WATER WAVE EVOLUTION

The nearshore characteristics of the closure approximation is strongly dependent on the dissipation characteristics. Thus to complete our nearshore wave model based on the QNR closure approximation (equations (6) and (9)), we adopt a quadratic weighting of the dissipation (Chen et al. 1997; Kaihatu et al. 2007; Pak 2011) and write the dissipation term in (6) as

$$\nu_1 = \omega^2 \frac{\epsilon}{2m_2} \tag{14}$$

Here ϵ is the total (frequency-integrated) rate of wave dissipation associated with depth-

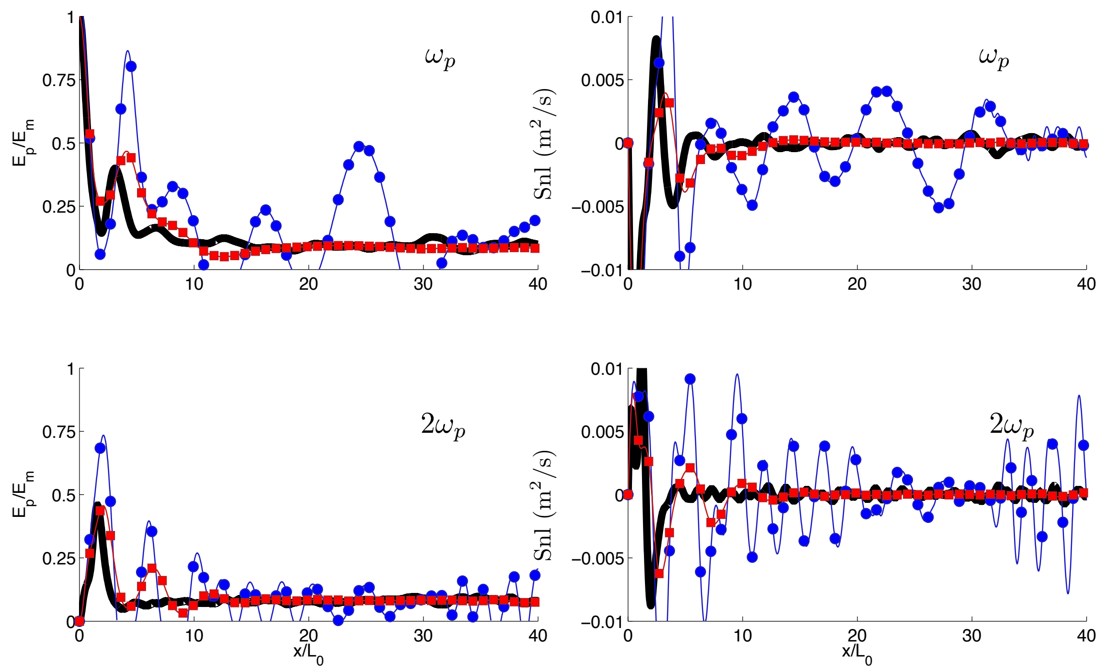

FIG. 2. Left panels: comparison normalized energy at peak frequency (top panels) and first harmonic (bottom panels). Right panels show the nonlinear energy transfer (Snl) at the peak frequency (top) and first harmonic (bottom). Shown are Monte Carlo predictions (black solid line without markers), QN approximation (blue line with circle markers), and QNR approximation (red line with square markers).

induced breaking, estimated using the expressions in Janssen and Battjes (2007), and $m_2 = \int \omega^2 \mathcal{E}\, d\omega$.

One-dimensional wave propagation: laboratory observations

We will compare our model to observations made by Boers (1996) who conducted experiments in a 40-m-long, 0.8-m-wide wave flume. The flume was equipped with a hydraulically driven, piston-type wave generator. The bottom profile used in the experiments mimics a barred sandy beach (see figure 3), with the origin of the x-axis at the beginning of the slope, and the dots at SWL (in figure 3) indicating the 70 locations where wave observations are available.

We consider a case with moderately steep incident waves (incident wave $H_s = 10\,\mathrm{cm}$, $T_p = 3.33\,\mathrm{s}$; referred to as case $1C$ in Boers 1996). The model is initiated with the observations at $x = 0$, we set the breaker index to $\gamma = 0.85$ (based on some trial runs to calibrate dissipation for wave heights), and the relaxation constant is set to $\beta = 1.8$ (estimated from a regression analysis of the observed relaxation characteristics, see Pak 2011). The nonlinear dynamics are not sensitive to variations in γ (which controls the bulk dissipation rate) but are affected by variations in frequency weighting of dissipation (quadratic) and the strength of relaxation controlled by β. The model-predicted spectra at $x = 20\,\mathrm{m}$ and $x = 25\,\mathrm{m}$ are in very good agreement overall with what is observed (see figure 4), including the slope of the tail. Although the simulated spectra do not resolve all the details in the observed spectral shape (the modeled spectra are more smooth),

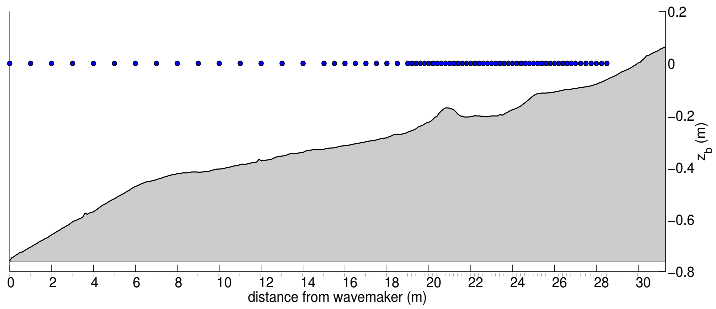

FIG. 3. Beach profile Boers 1996, positive x-direction from left to right. Circles at SWL and x-axis ticks indicate wave gauge positions.

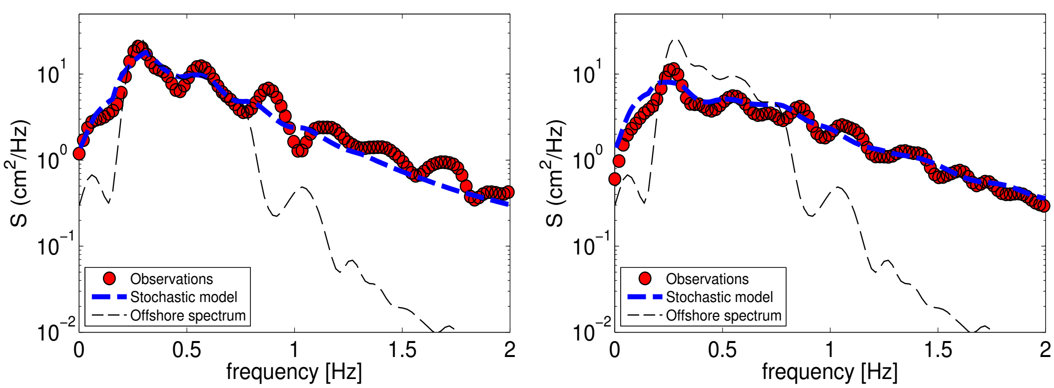

FIG. 4. Observed (circles) and predicted (thick dashed line) spectra at $x = 20\,\mathrm{m}$ (left panel) and $x = 25\,\mathrm{m}$ (right panel). The thin dashed line represents the observed spectrum at $x = 0\,\mathrm{m}$.

the over-prediction of energy in the spectral tail (which we saw in the comparison to conservative Monte Carlo simulations) is not observed.

The observed (from bispectral analysis) and modeled space-frequency structure of nonlinear energy transfers in good agreement overall (figure 5). However, some differences remain at higher frequencies ($1\,\mathrm{Hz} < f < 1.5\,\mathrm{Hz}$) over the breaker bar (near $x \approx 20\,\mathrm{m}$) and on the slope shoreward of the trough ($x \approx 24\,\mathrm{m}$), where energy transfers are somewhat underestimated by the model. The bulk third-order statistics (skewness and asymmetry) are in good agreement with what is observed, although skewness values are slightly underestimated everywhere (figure 6).

CONCLUSIONS

We have presented a new, quasi-empirical closure approximation for the nonlinear evolution of wave statistics in shoaling gravity waves. The addition of a relaxation term to de-correlate triads in regions of strong nonlinearity, results in a more realistic representation of the nonlinear dynamics in shoaling gravity waves. Comparison to laboratory observations suggests that the model accurately captures the principal nonlinear dynamics in the surf zone, although nonlinear transfers to harmonic frequencies remain somewhat underestimated in regions of intense dissipation.

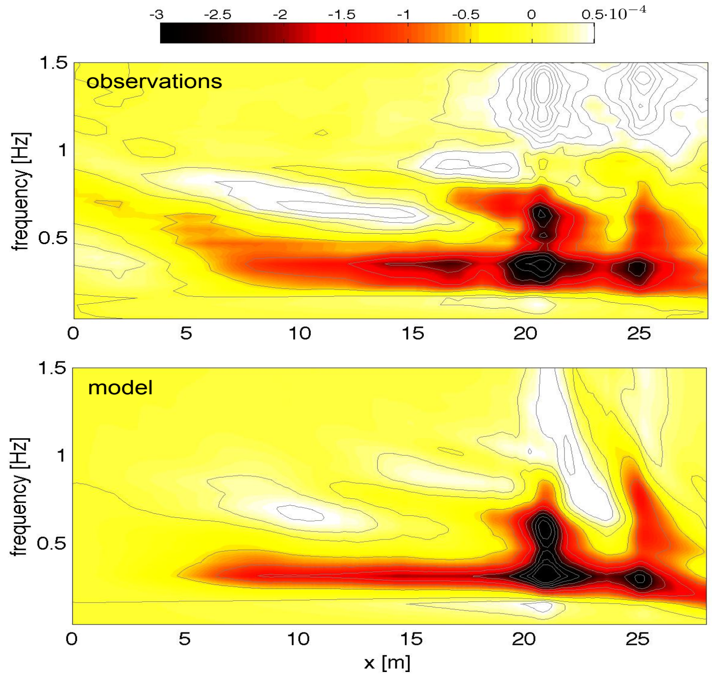

FIG. 5. Comparison of observed (top panel) and modeled (bottom panel) nonlinear energy transfers across the beach.

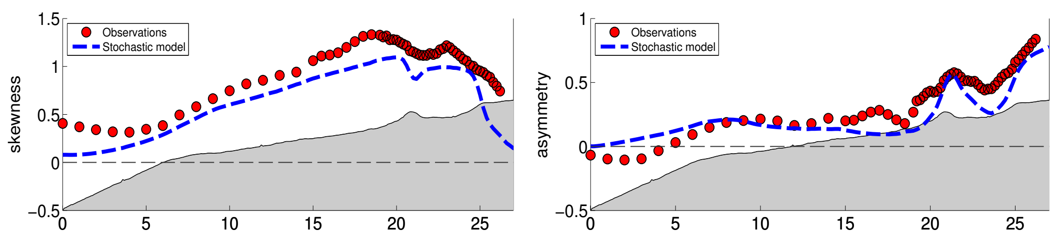

FIG. 6. Comparison of observed (circles) skewness (left panel) and asymmetry (right panel) and model results (thick dashed line).

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