New wind input term through experimental, theoretical and numerical consideration

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Abstract

We offer a new method for determination of the wind input term S_{in} responsible for flux of energy and momentum transfer to the wind-driven sea. This method is based on analysis of experimental data collected at different sites and their comparison with analytical solutions of Hasselmann equation.The validity of new wind input term is confirmed through comparison of the results of numerical simulation of Hasselmann equation with experimental data.

1 Introduction

Today, the vast majority of oceanographers believe in satisfactory description of the wind-driven sea by kinetic Hasselmann equation for wave action $N = N(\vec{k}, \vec{r}, t)$ (see, for instance [1]):

$$\frac{dN}{dt} = \frac{\partial N}{\partial t} + \frac{\partial \omega_k}{\partial \vec{k}} \frac{\partial N_k}{\partial \vec{r}} = S_{nl} + S_s \quad (1)$$

where $\omega_k = \sqrt{gk}$ – dispersion law for gravity waves, S_{nl} is nonlinear interaction term, $S_s = S_{in} + S_{diss}$ is the source term responsible for energy input from wind and dissipation due to white capping and interaction with turbulence.

The S_{nl} term is completely known. This is a complicated nonlinear integral operator possessing deeply hidden symmetry. The ”conservative” kinetic equation

$$\frac{dN}{dt} = S_{nl} \quad (2)$$

preserves the total wave action and formally preserves energy and momentum. This is not a real conservation – at any given initial data in a finite time equation Eq. (2) forms Kolmogorov-like tails carrying energy and momentum to the area of small wave numbers [4].

On the contrary, our knowledge about S_s is poor. Creation of reliable, well justified theory of S_{in} is hindered by strong turbulence of the air boundary layer over the sea surface. Even the most crucial for this theory vertical profile of the mean horizontal wind velocity is not properly known.

The data of direct measurements of S_{in} are scarce. As a result, we currently have at least dozen of heuristic models of S_{in}. A scatter between different models is large.For instance, the Donelan model [6] predicts S_{in} approximately five times higher than Hsiao-Shemdin model [12]. Comparison of different models of S_{in} is presented in the article [13].

It is pity, but understanding of dissipation term S_{diss} is not better. The theory is not developed, the experimental data are far from being complete. The forms of S_{diss} used in operational models are heuristic and badly justified.

The situation is aggravated by widely distributed opinion that in a real sea the nonlinear term has the same order of magnitude as the source term, i.e. $S_s \sim S_{nl}$. This viewpoint was formulated by O.Phillips [3] in 1985 and since this time is commonly accepted. Some authors (Donelan at al., [7, 11]) express even more radical opinion, assuming that S_{nl} is negligibly small with respect to S_s. According to this opinion, the wind-driven instability of gravity waves is arrested by wave-breaking. These speculative theories are supported neither by the theory, nor by experimental data.

In reality the nonlinear term S_{nl} surpasses the source term S_s at least by the order of magnitude. Let us formulate this statement in more accurate terms. It is reasonable to present S_s in a simple form

$$S_s = \gamma(k) N(\vec{k}, \vec{r}, t) \tag{3}$$

At small k, $\gamma(k) > 0$ and the source term is pumping the surface waves due to Cherenkov-type instability. For large k, $\gamma(k) < 0$. In the spectral range decrement of damping $\gamma(k)$ depends also on overall wave steepness $\mu \simeq (\nabla \eta^2)^{1/2}$, but we will not discuss this question in this article. The nonlinear term S_{nl} is given by the following expression:

$$\begin{aligned} S_{nl} &= \pi \int |T_{kk_1k_2k_3}|^2 \delta_{k+k_1-k_2-k_3} \delta_{\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3}} \\ &\quad (N_{k_1}N_{k_2}N_{k_3} + N_kN_{k_2}N_{k_3} - N_kN_{k_1}N_{k_2} - N_kN_{k_1}N_{k_3})\, dk_1dk_2dk_3 \end{aligned}$$

S_{nl} can be naturally split in two parts:

$$S_{nl} = F_k - \Gamma_k N_k \tag{4}$$

Here

$$\Gamma_k = \pi \int |T_{kk_1k_2k_3}|^2 \delta_{k+k_1-k_2-k_3} \delta_{\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3}} \left(N_{k_1}N_{k_2} + N_{k_1}N_{k_3} - N_{k_2}N_{k_3} \right) dk_1 dk_2 dk_3 \tag{5}$$

is the effective damping of gravity waves due to nonlinear wave interaction. Our recent numerical experiments [8], [10] show that for almost all situations $\Gamma_k >> \gamma_k$ and nonlinear dissipation essentially surpass the instability which is arrested not by white capping, but nonlinear wave interaction. In a typical wind-driven sea the forcing term F_k and the dissipation term $\Gamma_k N$ almost compensate each other, while S_s remains small, but very important correction.

The dominance of S_{nl} is indirectly supported by the following facts. Both in fetch-limited and duration-limited cases the dependencies of total energy and mean frequency of a wind-driven sea on duration and fetch are given by power-like functions. This fact can be explained in one simple way: as far as S_{nl} is the dominating term, one can describe the wind-driven case in the first approximation by conservative equation Eq. (2). It is clear from the beginning that this description is not complete. We should expect that this equation has a vast variety of solutions.

A natural class of them consists of self-similar solutions, which are studied both for the duration limit case $\frac{dN}{dx} = 0$ and for the fetch-limited case $\frac{dN}{dt} = 0$. In both cases self-similar solutions depend on two arbitrary parameters, which should be found by matching with the source term.

This procedure can be compared with derivation of the hydrodynamic equations from the classical Boltzmann kinetic equation. A general solution of the stationary homogeneous Boltzmann equation depends on three free parameters – density, mean velocity and temperature. Macroscopic equation imposed on these quantities composes the system of hydrodynamic equation.

Similar program for the Hasselmann equation was announced in our previous paper [5], but so far was not realized. In this article we modify this idea for determination of S_{in} from experimental data. We will exploit the fact that the sourced Hasselmann equation for a certain special form of S_s has unique self-similar solution, which can be compared with experimental data. This comparison makes possible to determine S_{in}.

2 Theoretical approach – self-similar solution

Self-similar solutions of conservative kinetic equation (2) were studied in the articles [1], [13]. In this chapter we study self-similar solutions of the forced kinetic equation

$$\frac{\partial \epsilon(\omega, \theta)}{\partial \theta} = S_{nl} + \gamma(\omega, \theta)\epsilon(\omega, \theta) \tag{6}$$

Here $\epsilon(\omega,\theta) = \frac{2\omega^4}{g}N(\vec{k},\theta)$, $k = \frac{\omega^2}{g}$ is the energy spectrum. We will not use the detailed structure of S_{nl}. It is enough to mention that by dimensional consideration

$$S_{nl} \simeq \omega \left(\frac{\omega^5 \epsilon}{g}\right)^2 \epsilon \tag{7}$$

As far as Eq.(6) formally conserve energy, one can present S_{nl} as follows:

$$S_{nl} = -\frac{\partial P}{\partial \omega} \tag{8}$$

The stationary solution $P = P_0 = const$ leads to the spectrum

$$\epsilon(\omega) = C_1 \frac{P_0^{1/3}}{\omega^4} \tag{9}$$

Here P_0 – energy flux to high wave numbers and C_1 is the Kolmogorov constant. Our recent numerical calculation give $C_1 \simeq 4\pi \cdot 0.219 = 2.75$ [8].

Eq.(6) has a self-similar solution if

$$\gamma(\omega,\theta) = \alpha\omega^{1+s} f(\theta) \tag{10}$$

Looking for self-similar solution in the form

$$\epsilon(\omega, t) = t^{p+q} F(\omega, t^q) \tag{11}$$

we find that

$$q = \frac{1}{s+1} \tag{12}$$

$$p = \frac{2q-1}{2} \tag{13}$$

Function $F(\xi) \to \xi$ at $\xi \to \infty$ and has a maximum at $\xi \sim \xi_0$. Thereafter we assume that $s < 2$. In this case we can assume that $F(\xi \simeq \xi^{-4}$ at $\xi \to \infty$. In other words, energy spectrum has Zakharov-Filonenko tail $\epsilon(\omega) \sim \omega^{-4}$. Total energy input is given by integral

$$P = \int_0^\infty \gamma(\omega,\theta)\epsilon(\omega,\theta) d\omega \tag{14}$$

This integral converges at $\omega \to \infty$, thus main energy input takes place in a neighborhood of the spectral maximum $\omega_0 \simeq \frac{\xi_0}{t^{p-3q}}$. For large ω

$$\epsilon \simeq \frac{t^{p-3q}}{\omega^4} \simeq \frac{t^{\frac{2-s}{2(s+1)}}}{\omega^4} \tag{15}$$

More accurately,

$$\epsilon(\omega,\phi) = \frac{\mu g u^{1-\xi} C_p^{\xi}}{\omega^4} \tag{16}$$

In Eq.(16) u is the wind velocity and $C_p = \frac{g}{\omega_p}$ is the phase velocity. As far as $\omega_p \simeq t^{-q}$, $C_p \simeq t^q$, $t \simeq C_p^{1/q}$,

$$\epsilon \simeq \frac{C_p^{\frac{p-3q}{q}}}{\omega^4} \tag{17}$$

In other words

$$\xi = \frac{p-3q}{q} = \frac{3q-1}{2q} \tag{18}$$

Remembering that $q = \frac{1}{s+1}$, we get $\xi = \frac{2-s}{2}$. We ended up with the following result:

$$\epsilon(\omega,\phi) \simeq \frac{\mu g U^{1-\xi} C_p^{\xi}}{\omega^4} g(\phi), \quad \mu \simeq 6 \cdot 10^{-3} \tag{19}$$

$$\xi = \frac{2-s}{2} \tag{20}$$

Now supposing $s = 4/3$ and $\gamma \simeq \omega^{7/3}$, we get $\xi = 1/3$, which is exactly experimental regression line prediction.

Because it is known from regression line Fig.1 that $\xi = 1/3$, we immediately get $s = 4/3$ and the wind input term

$$S_{wind} \simeq \omega^{7/3} \tag{21}$$

One should note that dependence (21) has already been predicted by Resio and Perrie [18] from dimensional consideration.

3 Experimental evidence

The structure of wind input term found from self-similar consideration is just a conjecture. To check its relation to reality, we are looking first at the experimental evidence presented by Resio at al. [15] through analysis of the data sets from multiple experimental installations scattered around the world.

To understand the relation of experimental evidence and theoretical results from the previous section, one should notice another notations for the spectrum (9) used in [15]:

$$E_4(\omega) = \frac{2\pi\alpha_4 u g}{\omega^4}, \quad F_4(k) = \beta k^{-5/2} \tag{22}$$

where $\beta = \frac{1}{2}\alpha_4 u g^{-1/2}$.

These notations are based on relation of spectral density $E(f)$ and $F(k)$ in frequency $f = \frac{\omega}{2\pi}$ and wave-number k bases:

$$F(k) = \frac{c_g}{2\pi} E(f) \tag{23}$$

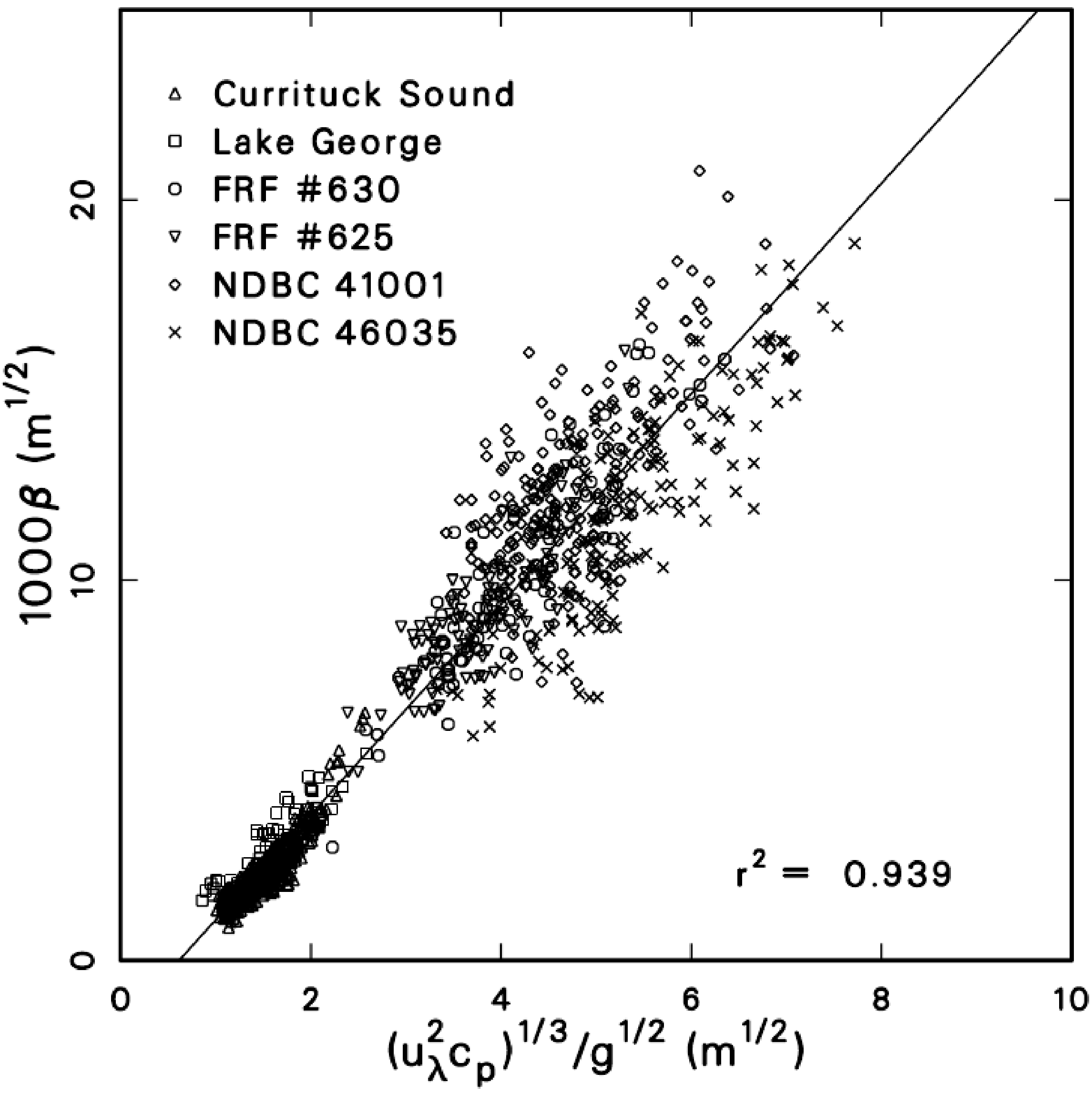

Figure 1: *Correlation of equilibrium range coefficient β with $(u_\lambda^2 c_p)^{1/3}/g^{1/2}$ based on data from six disparate sources. Adopted from [15]*

where $c_g = \frac{d\omega}{dk}$ is group velocity and $\omega = \sqrt{gk}$ is the linear dispersion.

The experimental data [15] show that energy spectrum $F(k)$ estimated through averaging $< k^{5/2}F(k) >$, can be approximated by linear regression line as a function of $(u_\lambda^2 c_p)^{1/3} g^{-1/2}$. Fig.1 shows that the regression line

$$\beta = \frac{1}{2}\alpha_4 \left[(u_\lambda^2 c_p)^{1/3} - u_0 \right] g^{-1/2} \tag{24}$$

indeed, seems to be a reasonable approximation of these observations. Here $\alpha_4 = 0.00553$, $u_0 = 1.93$ m/sec, c_p is the spectral peak phase speed and u_λ is the wind speed at the elevation equal to a fixed fraction $\lambda = 0.065$ of the spectral peak wavelength $2\pi/k_p$, where k_p is the spectral peak wave number. Resio at al. [15] assume that all winds to follow neutrally stratified logarithmic profile

$$u_\lambda = \frac{u_\star}{\kappa} \ln \frac{z}{z_0} \tag{25}$$

having Von Karman coefficient $\kappa = 0.41$, where $z = \lambda \cdot 2\pi/k_p$ is the elevation equal to a fixed fraction $\lambda = 0.065$ of the spectral peak wavelength $2\pi/k_p$, where k_p is the spectral peak wave number, and $z_0 = \alpha_C u_\star^2/g$ subject to Charnok 1955 [17] surface roughness with $\alpha_C = 0.015$.

4 Numerical simulation

To check out self-similar conjecture (21) we performed numerical simulation of Hasselmann equation

$$\frac{\partial n_{\vec{k}}}{\partial t} = S_{nl} + S_{wind} + S_{diss} \tag{26}$$

with new input term taken in the form:

$$S_{wind} = 0.2 \frac{\rho_{air}}{\rho_{water}} \omega \left(\frac{\omega}{\omega^\star} \right)^{4/3}, \omega^\star = \frac{g}{u^\star}, \quad \frac{\rho_{air}}{\rho_{water}} = 1.3 \cdot 10^{-3} \tag{27}$$

This model also needs to be supplied with the dissipation term S_{diss}, which is explicitly unknown, but can be taken into account in some way. It was proposed by Resio [16], that white-capping dissipation term S_{diss} can be introduced implicitly through energy spectral tail proportional to f^{-5} and stretching from $f_d = 1.1$ to $f_{max} = 2.0$. To date, this approach is confirmed by both experimental observations [16] and numerical experiments, providing effective direct cascade sink for energy entering the wave system from the wind input.

Typical picture of the directional energy spectrum for the case $u = 10$ m/sec is presented on Fig.2. Once can distinct three separate areas of the spectrum – areas of spectral peak, intermediate portion of the spectral tail proportional to f^{-4} and high-frequency portion of the spectrum, proportional to f^{-5}.

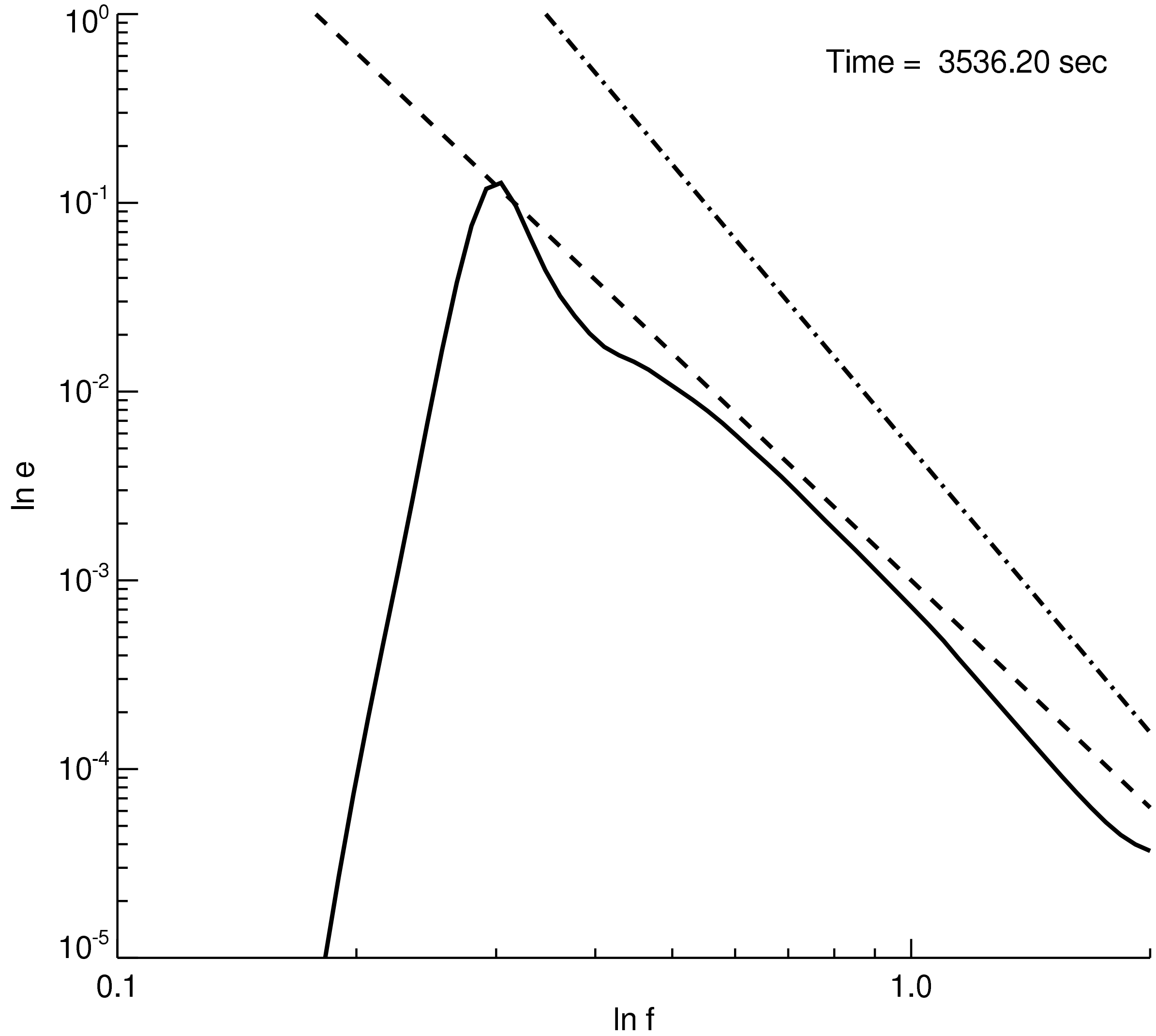

Figure 2: *Typical picture of* $\ln \epsilon(f)$ *as a function of* $\ln f$*, wind speed* $u = 10.0$ *m/sec. Solid line – directional spectrum, dashed line – spectrum* f^{-4}*, dash-dotted line – spectrum* f^{-5}.

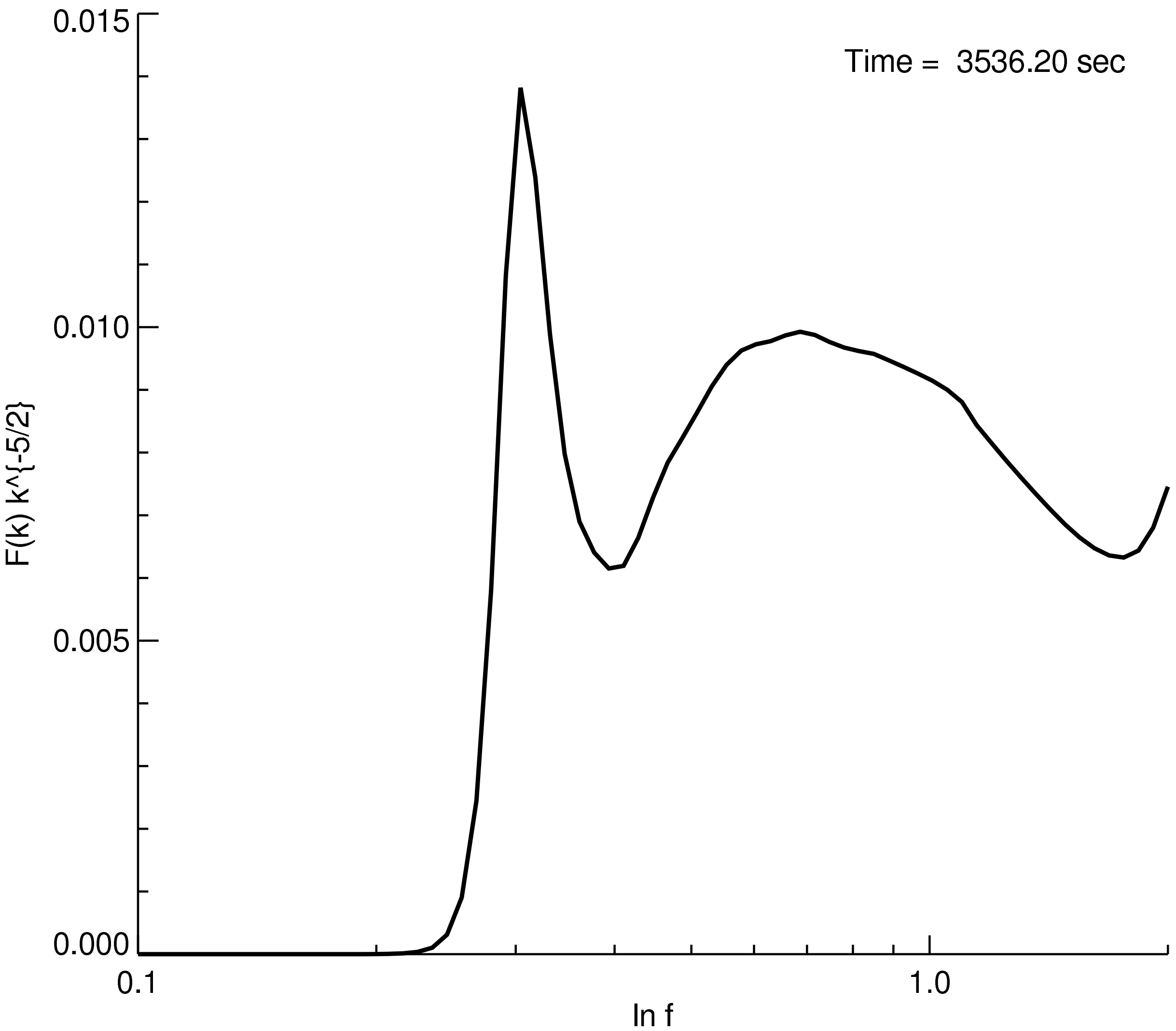

Figure 3: *Compensated spectrum* $F(k)k^{5/2}$ *as a function of* $\ln f$*, wind speed* $u = 10.0$ *m/sec*

Compensated spectrum $F(k) \cdot k^{5/2}$ is presented on Fig.3. One can see plateu-like region responsible for $k^{-5/2}$ behavior, equivalent to f^{-4} tail in frequency presentation. This exact solution of Eq.26 was found by Zakharov and Filonenko in 1966 [9].

Now we are turning to direct comparison of the numerical simulation results of Eq.26 – 27 with the experimental results collected by Resio at al. [15] on the graph Fig.(1).

Fig. 4 presents the plot of function $\beta = F(k) \cdot k^{5/2}$ as a function of parameter $(u_\lambda^2 C_p)^{1/3}/g^{1/2}$ for four different runs, corresponding to wind speeds $u = 2.5, 5.0, 10.0, 20.0$ and show good correspondence with the regression line for values of $u = 2.5, 5.0, 10.0$. The results corresponding to $u = 20.0$ are a bit off the regression line, but exhibits the same slope.

Another important relationship can be derived from joint consideration of Eqs. (19), (22) and (23):

$$1000\beta = 3\frac{(u^2 c_p)^{1/3}}{g^{1/2}} \tag{28}$$

The plot of relationship Eq.(28) on Fig. 4 shows close behavior of theoretical, experimental and numerical considerations.

5 Conclusion

We offer theoretical explanation to experimental regression line, obtained from experimental data from 6 independent sites. It consists in existence of self-similar solution of Hasselmann equation supplied with specific wind input term. For this form of wind input term we observe good correspondence of experimental, theoretical and numerical results in wide range of wind speeds. The new form of wind input term can improve the quality of ocean waves operational models forecasts.

6 Acknowledgments

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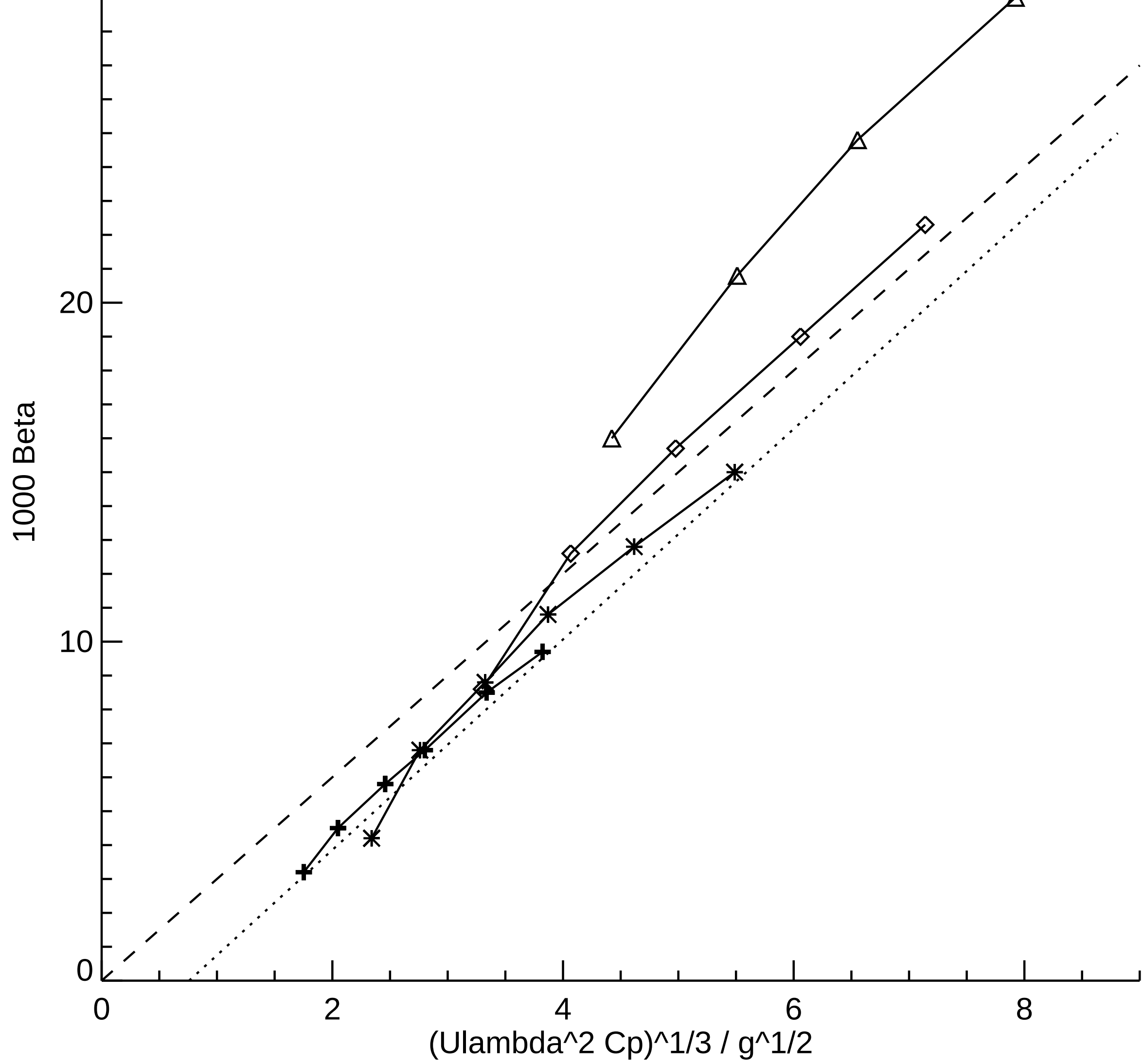

Figure 4: Experimental, theoretical and numerical evidence on the single graph for 1000β as a function of $(u_\lambda^2 c_p)^{1/3}/g^{1/2}$. Experimental result: dotted line – experimental regression line from Fig.1. Theoretical result: dashed line – theoretical relation Eq.(28). Numerical results: crosses correspond to $u = 2.5$, stars to $u = 5.0$, rectangles to $u = 10.0$, triangles to $u = 20.0$