Coherent Interference and Diffraction in Random Waves

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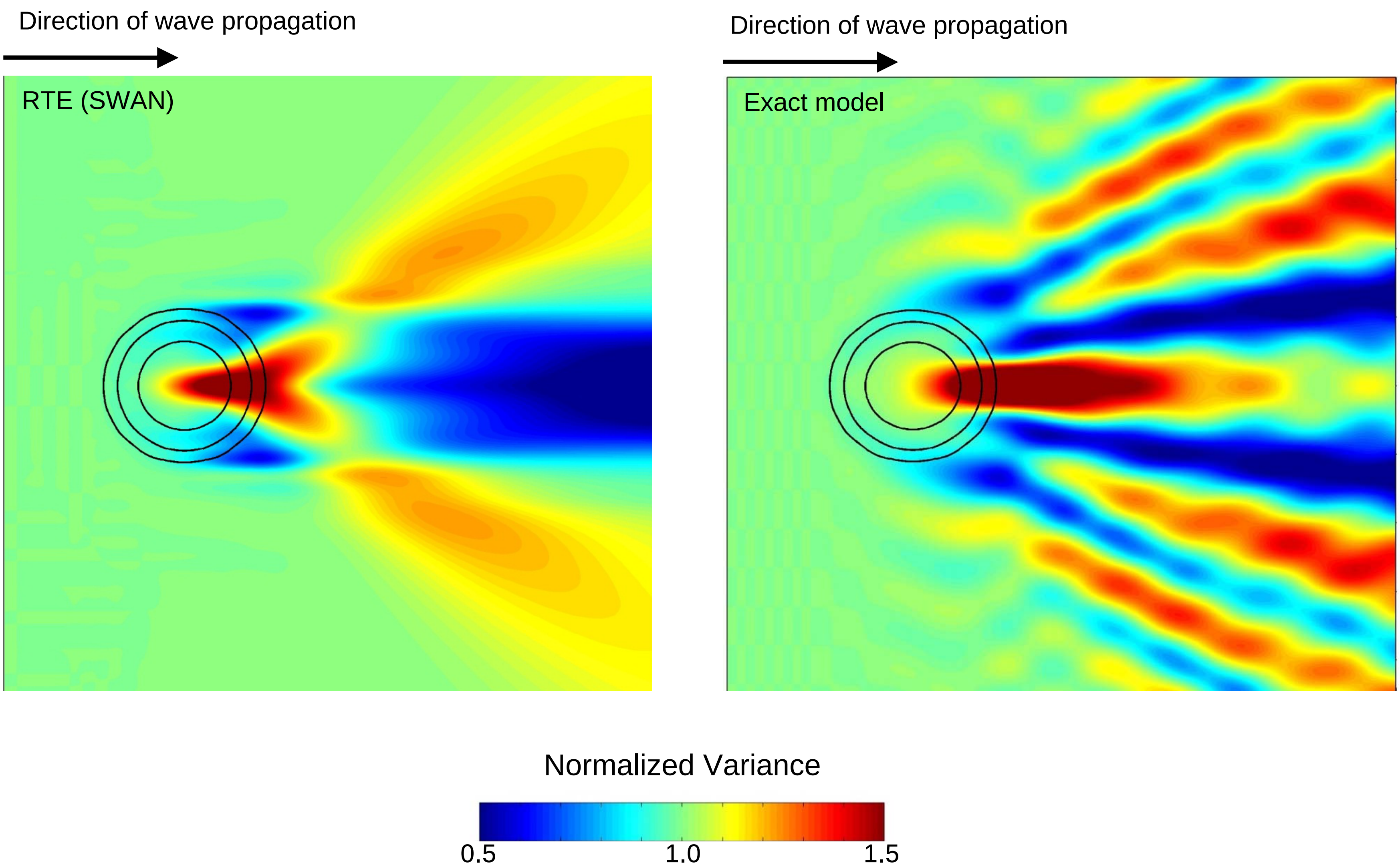
Outline

- What is wrong with the RTE?
- How do we improve it?
- Examples
- Concluding remarks

Radiative transport equation (RTE)

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla_{\boldsymbol{k}} \Omega \cdot \nabla_{\boldsymbol{x}} \mathcal{E} - \nabla_{\boldsymbol{x}} \Omega \cdot \nabla_{\boldsymbol{k}} \mathcal{E} = S$$

What is wrong with the RTE`?



What is wrong with the RTE?

Consider a two-component wavefield described by:

$$\zeta(\boldsymbol{x},t) = \widehat{\zeta}_1 \exp\left[i\Phi_1\right] + \widehat{\zeta}_2 \exp\left[i\Phi_2\right] \qquad \Phi_j = \boldsymbol{k}_j \cdot \boldsymbol{x} - \omega_j t$$

Variance:

$$\langle \zeta\zeta^* \rangle = \left\langle \widehat{\zeta}_1 \widehat{\zeta}_1^{\,*} \right\rangle + \left\langle \widehat{\zeta}_2 \widehat{\zeta}_2^{\,*} \right\rangle + \left(\left\langle \widehat{\zeta}_1 \widehat{\zeta}_2^{\,*} \right\rangle \exp\left[i\Phi_1 - i\Phi_2\right] + C.C. \right)$$

auto-variance contributions

cross-variance contributions

How do transport these interference terms?

Evolution of interference contributions

Consider the transform pair for the wave variable ζ :

$$\zeta(\boldsymbol{x},t) = \int \widehat{\zeta}(\boldsymbol{k},t)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\,d\boldsymbol{k},$$

$$\widehat{\zeta}(\boldsymbol{k},t) = \frac{1}{(2\pi)^2}\int \zeta(\boldsymbol{x},t)\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\,d\boldsymbol{x}.$$

Local dispersion relation in slowly changing medium:

$$\omega = \Omega(\boldsymbol{k},\boldsymbol{x},t)$$

Using operator correspondence

$$-\mathrm{i}\omega \rightarrow \partial_t, \qquad \boldsymbol{x} \rightarrow \mathrm{i}\nabla_{\boldsymbol{k}},$$

$$\boxed{\partial_t\widehat{\zeta}(\boldsymbol{k},t) = -\mathrm{i}\Omega(\boldsymbol{k},\mathrm{i}\nabla_{\boldsymbol{k}},t)\widehat{\zeta}(\boldsymbol{k},t)}$$

Evolution of interference contributions

A transport equation for $\gamma(\boldsymbol{k}_1, \boldsymbol{k}_2, t) = \left\langle \widehat{\zeta}(\boldsymbol{k}_1, t)\widehat{\zeta}^*(\boldsymbol{k}_2, t) \right\rangle$

$$\left[\partial_t + \mathrm{i}\Omega(\boldsymbol{k}_1, i\nabla_{\boldsymbol{k}_1}, t) - \mathrm{i}\Omega\left(\boldsymbol{k}_2, -\mathrm{i}\nabla_{\boldsymbol{k}_2}, t\right)\right]\gamma(\boldsymbol{k}_1, \boldsymbol{k}_2, t) = 0$$

- Introduce: $\boldsymbol{k} = \frac{1}{2}\left(\boldsymbol{k}_1 + \boldsymbol{k}_2\right), \quad \boldsymbol{u} = \boldsymbol{k}_1 - \boldsymbol{k}_2,$
- take the inverse $\boldsymbol{u} \to \boldsymbol{x}$ transform

$$\partial_t \mathcal{E} = -\mathrm{i}\left[\Omega\left(\boldsymbol{k} - \frac{\mathrm{i}}{2}\nabla_{\boldsymbol{x}}, \boldsymbol{x} + \frac{\mathrm{i}}{2}\nabla_{\boldsymbol{k}}, t\right) - \Omega\left(\boldsymbol{k} + \frac{\mathrm{i}}{2}\nabla_{\boldsymbol{x}}, \boldsymbol{x} - \frac{\mathrm{i}}{2}\nabla_{\boldsymbol{k}}, t\right)\right]\mathcal{E}$$

where

$$\mathcal{E}\left(\boldsymbol{k}, \boldsymbol{x}, t\right) = \frac{1}{2}\int \left\langle \widehat{\zeta}\left(\boldsymbol{k} + \frac{\boldsymbol{u}}{\boldsymbol{2}}, t\right)\widehat{\zeta}^*\left(\boldsymbol{k} - \frac{\boldsymbol{u}}{2}, t\right)\right\rangle \exp\left(\mathrm{i}\boldsymbol{u}\cdot\boldsymbol{x}\right)\, d\boldsymbol{u}$$

The Coupled Mode Spectrum

Consider $\mathcal{E}(\boldsymbol{k}, \boldsymbol{x}, t)$ (Coupled Mode Spectrum)

- Transform of the two point correlator: $\left\langle \widehat{\zeta}\left(\boldsymbol{k}+\frac{\boldsymbol{u}}{\mathbf{2}}, t\right) \widehat{\zeta}^{*}\left(\boldsymbol{k}-\frac{\boldsymbol{u}}{2}, t\right)\right\rangle$
- Marginal relation $\mathcal{V}(\boldsymbol{x}, t)=\int \mathcal{E}(\boldsymbol{k}, \boldsymbol{x}, t)\, d\boldsymbol{k}.$ represents Variance

However, it is not a variance density spectrum

- Captures complete second order statistics
- Is real, but not point-wise positive

Evolution of interference contributions

Governing equation

$$\partial_t \mathcal{E} = -\mathrm{i}\left[\Omega\left(\boldsymbol{k} - \frac{\mathrm{i}}{2}\nabla_{\boldsymbol{x}}, \boldsymbol{x} + \frac{\mathrm{i}}{2}\nabla_{\boldsymbol{k}}, t\right) - \Omega\left(\boldsymbol{k} + \frac{\mathrm{i}}{2}\nabla_{\boldsymbol{x}}, \boldsymbol{x} - \frac{\mathrm{i}}{2}\nabla_{\boldsymbol{k}}, t\right)\right]\mathcal{E}$$

Taylor series for the operators

$$\frac{\partial \mathcal{E}}{\partial t} + \sum_{n=1}^{\infty}\left[\sum_{i_1=1}^{4}\cdots\sum_{i_{2n-1}=1}^{4} \mathcal{C}_{i_1,\ldots,i_{2n-1}}\left(\prod_{m=1}^{2n-1}\frac{\partial}{\partial q_{i_m}}\right)\right]\mathcal{E} = 0$$

First order...

$$\boxed{\frac{\partial \mathcal{E}}{\partial t} + \nabla_{\boldsymbol{k}}\Omega \cdot \nabla_{\boldsymbol{x}}\mathcal{E} - \nabla_{\boldsymbol{x}}\Omega \cdot \nabla_{\boldsymbol{k}}\mathcal{E} = 0}$$

Radiative transport equation!

Quasi-Coherent Approximation

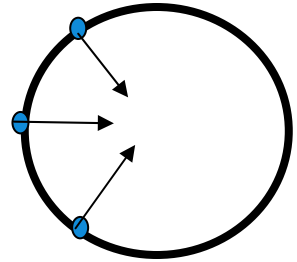
The Quasi-Coherent (QC) approximation (with $\boldsymbol{q} = [\boldsymbol{x}, \boldsymbol{k}]^T$)

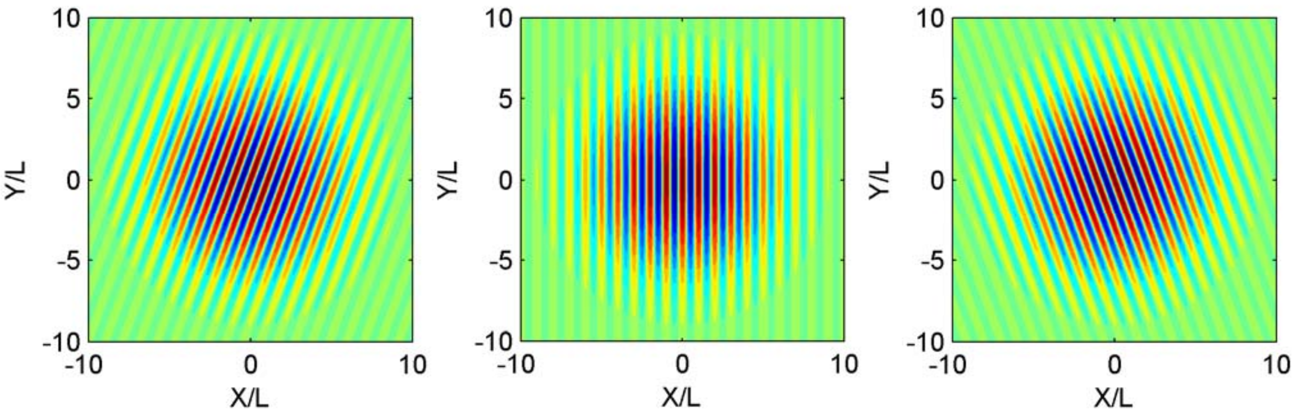
$$\left[\frac{\partial}{\partial t} + \mathcal{C}_i \frac{\partial}{\partial q_i} + \mathcal{C}_{i,j,k} \frac{\partial^3}{\partial q_i \partial q_j \partial q_k}\right] \mathcal{E} = 0$$

Rad. Transp. Equation

Corrections for coherent interference

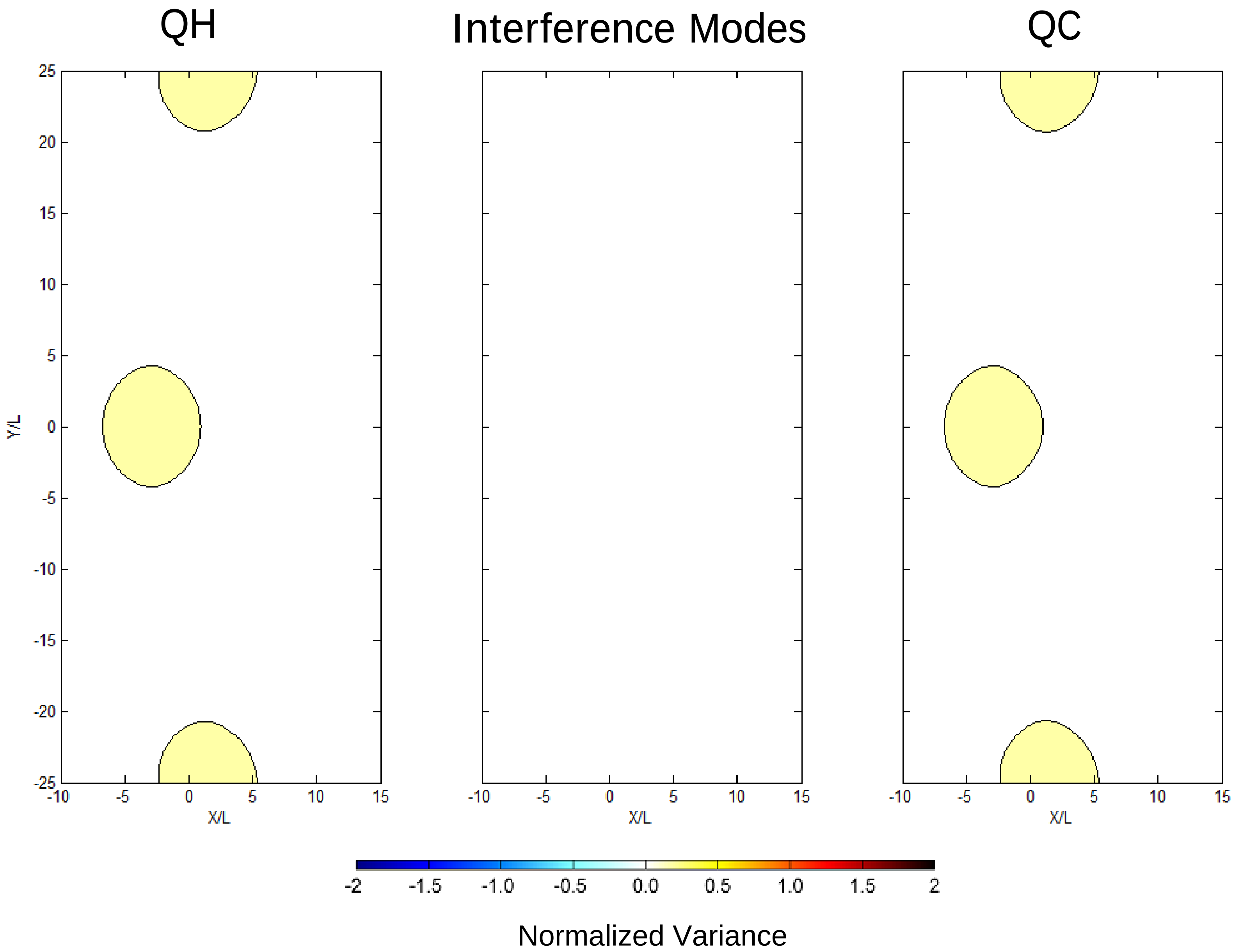
Example: Coherent Gaussian Packets





$$\zeta\left(\mathbf{x}, t=t'\right) = \sum_{m=1}^{N} A_m \exp\left(-\alpha\left|\mathbf{x}\right|^2 + i\mathbf{k}_m \cdot \mathbf{x}\right)$$

Example: Coherent Gaussian Packets



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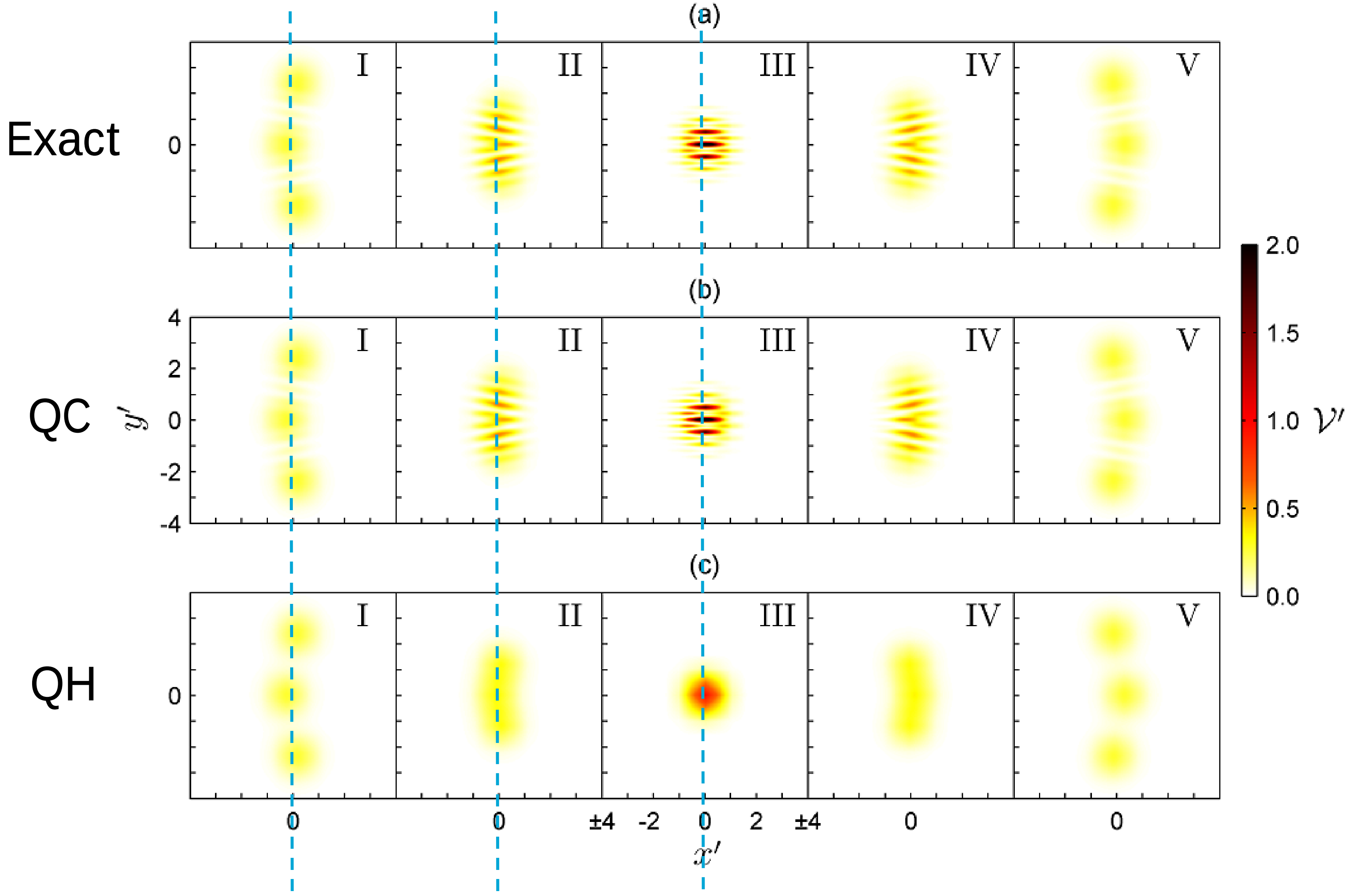

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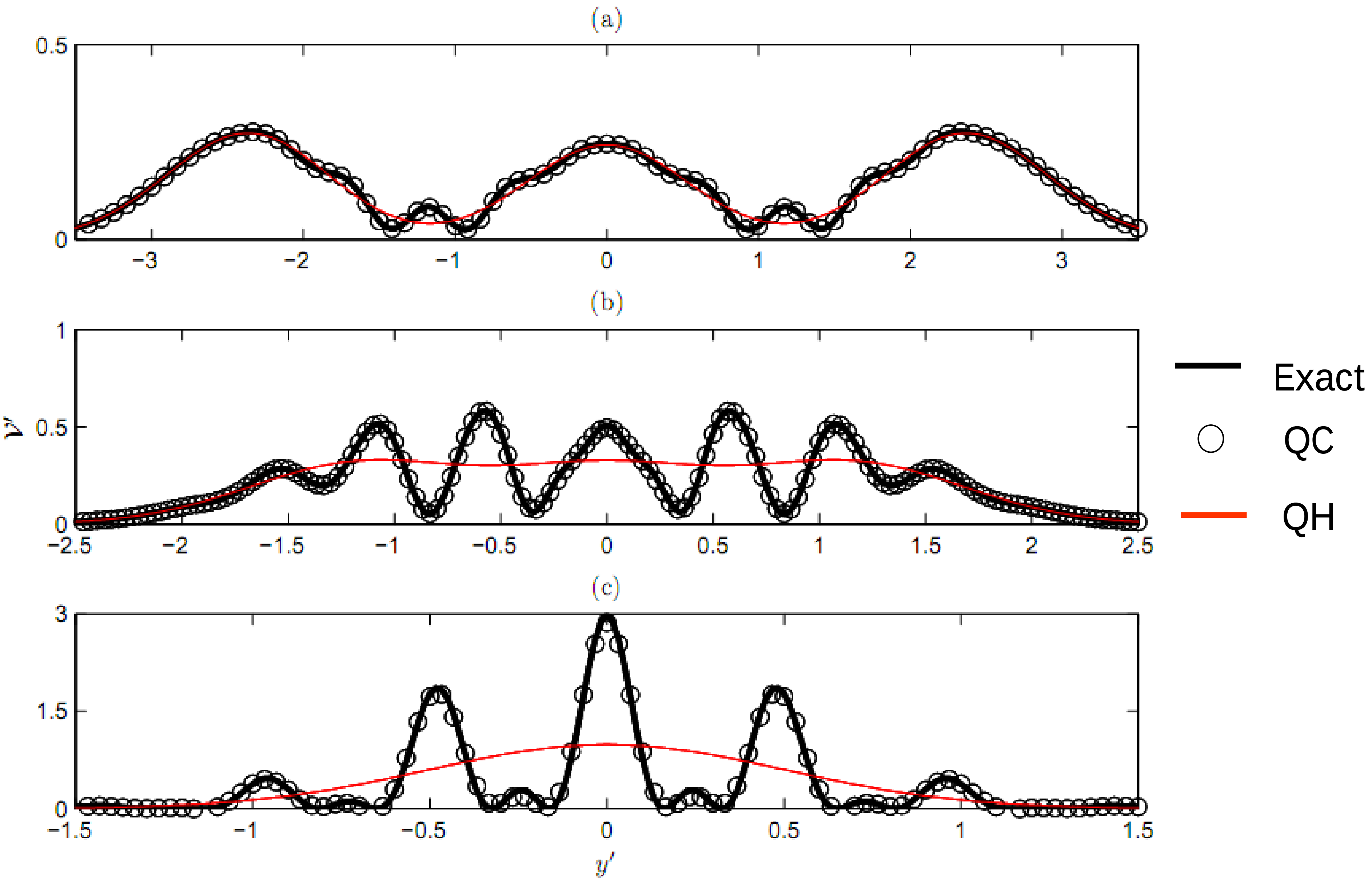

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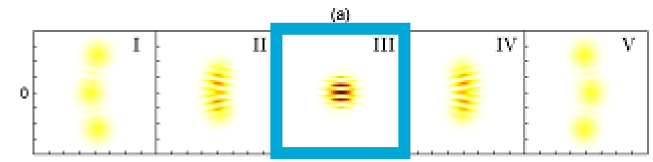
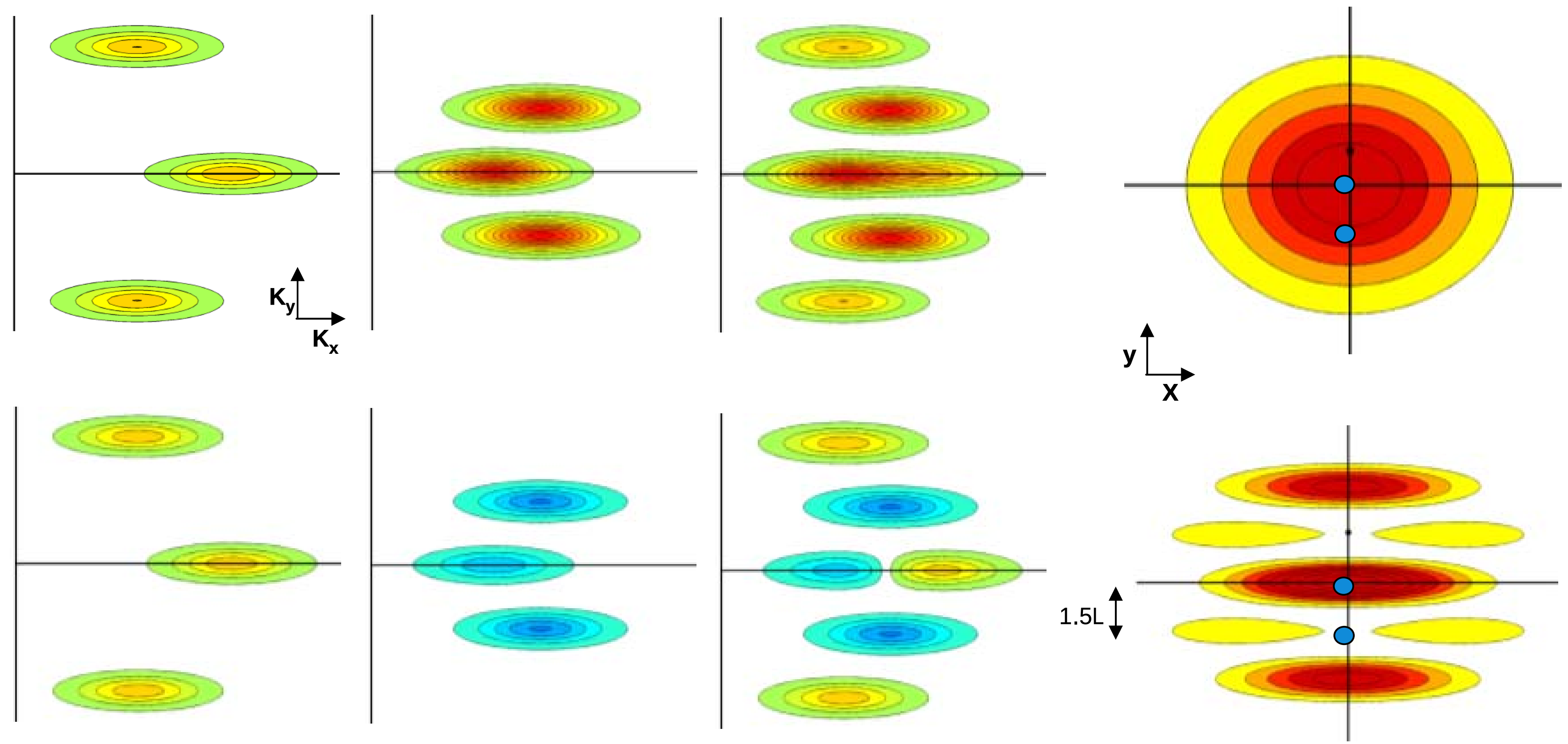
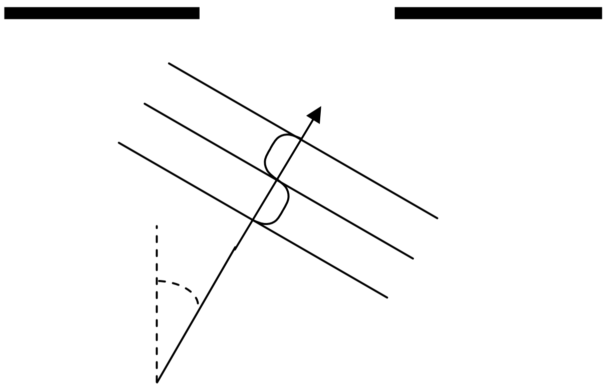


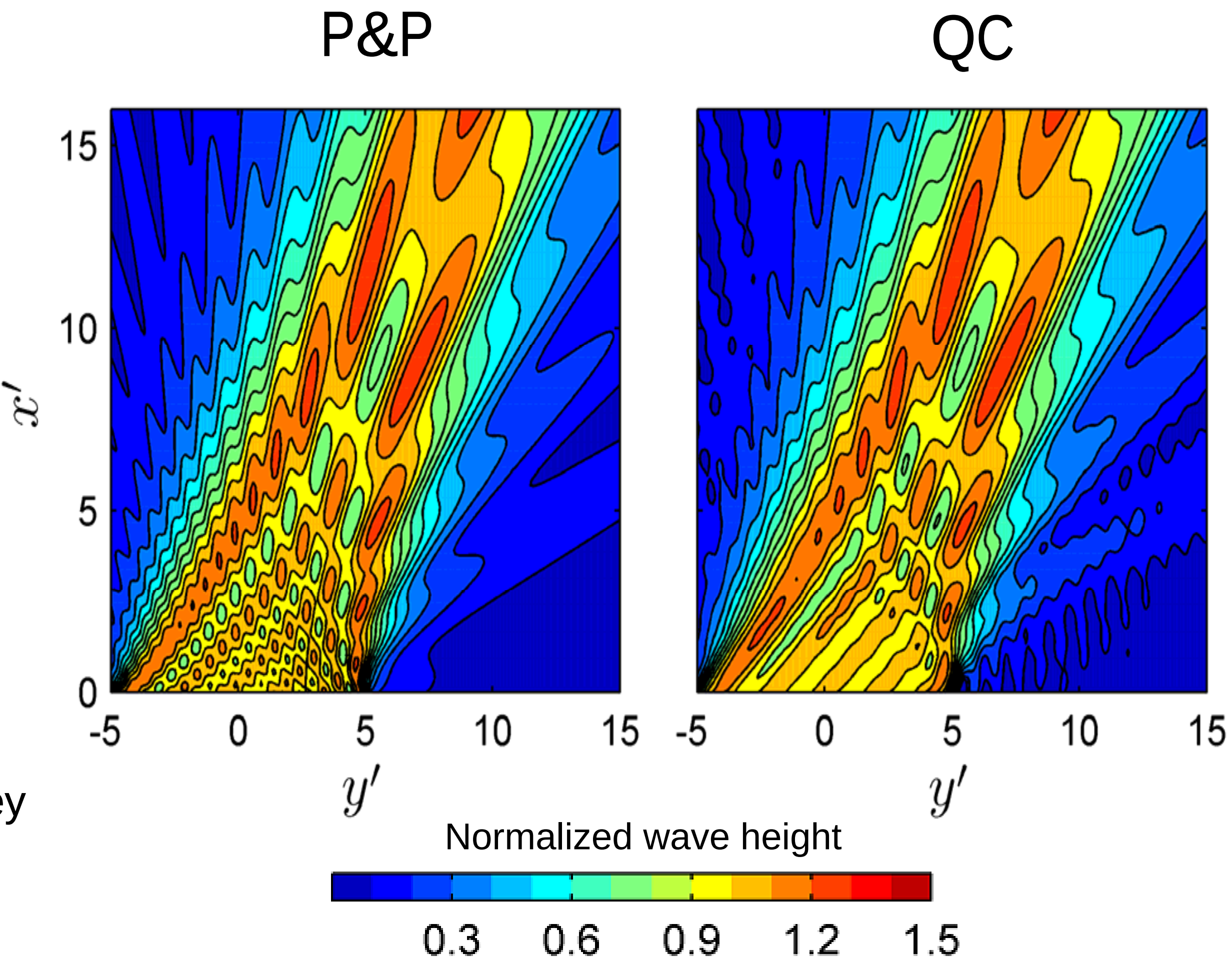
auto-modes Interference-modes CM-spectrum



Example: Diffraction through a gap

Wide angle Diffraction



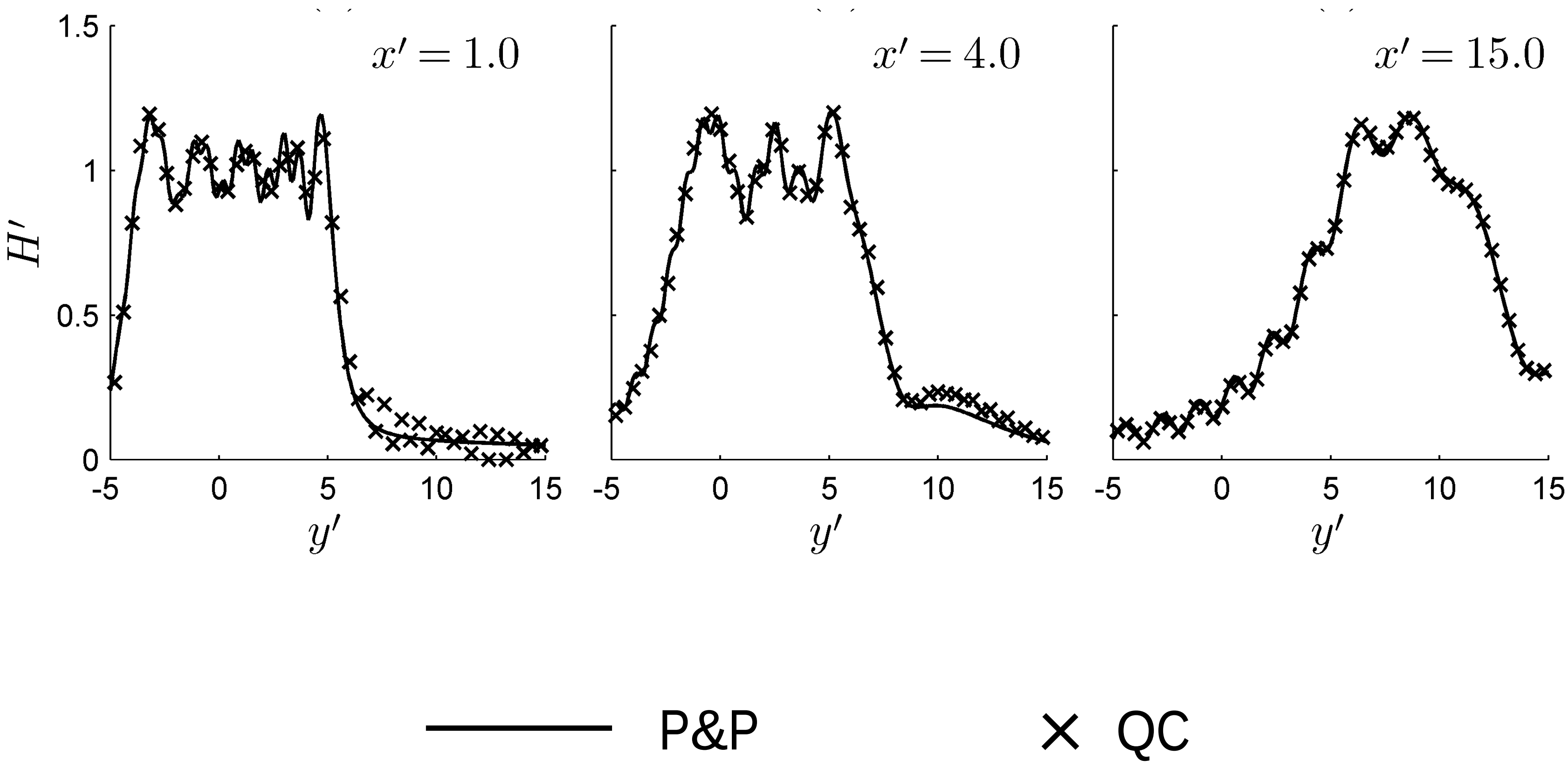


- Monochromatic wave

- Compared to solution by Penney & Price (1952)

Examples

Wide angle Diffraction



Concluding remarks

- Consistent extension of Quasi-Homogeneous theory to inhomogeneous fields
- Compatible with operational wave modeling framework makes coupling to regional models straightforward.
- Likely areas of improvement: coastal focal zones, sheltering (headlands), and river mouths and tidal inlets.
- Application to variable topography in progress
- Same approach can be used to include non-Gaussian effects in shallow water.