Poincare normal form of equations for nonlinear gravity waves

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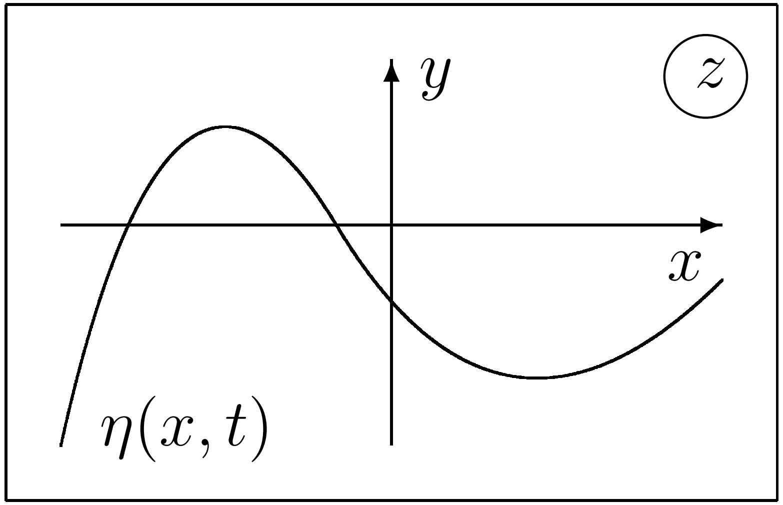
Equations



potential irrotational flow

$$\triangle\phi(x,y,t)=0$$

Boundary conditions: $$\left[\begin{array}{l}\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+g\eta=\frac{P}{\rho},\\ \frac{\partial\eta}{\partial t}+\eta_x\phi_x=\phi_y\end{array}\right| \text{at } y=\eta(x,t).$$

$$\begin{aligned}\frac{\partial\phi}{\partial y} &= 0, y\to-\infty,\\ \frac{\partial\phi}{\partial x} &= 0, |x|\to\infty.\end{aligned}$$

Hamiltonian

Hamiltonian H is the total energy of the fluid $H = T + U$

$$
\begin{aligned}
T &= \frac{1}{2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\eta} (\nabla\Phi)^2 dy\,, \\
U &= \frac{g}{2}\int \eta^2 dx\,.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\eta}{\partial t} &= \frac{\delta H}{\delta\Psi}\,, \\
\frac{\partial\Psi}{\partial t} &= -\frac{\delta H}{\delta\eta}\,, \\
\Psi(x,t) &= \Phi(x,y,t)|_{y=\eta}
\end{aligned}
$$

$$
H = \frac{1}{2}\int_{-\infty}^{\infty} \Psi\hat{G}(\eta)\Psi dx \quad + \quad \frac{g}{2}\int_{-\infty}^{\infty} \eta^2 dx
$$

Classical variables Ψ, η

Normal complex variable a_k:

$$\eta_k = \sqrt{\frac{\omega_k}{2g}}(a_k + a^*_{-k}) \qquad \psi_k = -i\sqrt{\frac{g}{2\omega_k}}(a_k - a^*_{-k}) \qquad \omega_k = \sqrt{gk}$$

$$\begin{aligned} \mathcal{H} &= \int \omega_k |a_k|^2 + \int V^k_{k_1 k_2}\{a^*_k a_{k_1} a_{k_2} + a_k a^*_{k_1} a^*_{k_2}\}\delta_{k-k_1-k_2} dk dk_1 dk_2 \\ &+ \frac{1}{3}\int U_{k k_1 k_2}\{a_k a_{k_1} a_{k_2} + a^*_k a^*_{k_1} a^*_{k_2}\}\delta_{k+k_1+k_2} dk dk_1 dk_2 + \\ &+ \frac{1}{4}\int W^{k k_1}_{k_2 k_3} a^*_k a^*_{k_1} a_{k_2} a_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 \end{aligned}$$

Normal variables a_k

a_k satisfies the equation

$$\frac{\partial a_k}{\partial t} + i\frac{\delta H}{\delta a_k^*} = 0,$$

Three wave resonances are absent

$$\begin{aligned} k &= k_1 + k_2, \\ \omega_k &= \omega_{k_1} + \omega_{k_2}, \end{aligned}$$

NO!

Cubic nonresonat terms can be excluded by canonical transformation:

$$a_k \longrightarrow b_k.$$

Transformation $a_k \to b_k$

$$\begin{aligned} a_k &= b_k + \int \Gamma^k_{k_1 k_2} b_{k_1} b_{k_2} \delta_{k-k_1-k_2} - 2\int \Gamma^{k_2}_{k k_1} b^*_{k_1} b_{k_2} \delta_{k+k_1-k_2} + \\ &+ \int \Gamma_{k k_1 k_2} b^*_{k_1} b^*_{k_2} \delta_{k+k_1+k_2} + \int B^{k k_1}_{k_2 k_3} b^*_{k_1} b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} + \ldots \end{aligned}$$

$$\begin{aligned} B^{k k_1}_{k_2 k_3} &= \Gamma^{k_1}_{k_2 k_1-k_2} \Gamma^{k_3}_{k k_3-k} + \Gamma^{k_1}_{k_3 k_1-k_3} \Gamma^{k_2}_{k k_2-k} - \\ &- \Gamma^{k}_{k_2 k-k_2} \Gamma^{k_3}_{k_1 k_3-k_1} - \Gamma^{k_1}_{k_3 k_1-k_3} \Gamma^{k_2}_{k_1 k_2-k_1} - \\ &- \Gamma^{k+k_1}_{k k_1} \Gamma^{k_2+k_3}_{k_2 k_3} + \Gamma_{-k-k_1 k k_1} \Gamma_{-k_2-k_3 k_2 k_3} + \tilde{B}^{k k_1}_{k_2 k_3}. \end{aligned}$$

$\tilde{B}^{k k_1}_{k_2 k_3}$ is arbitrary function, satisfying symmetry conditions:

$$\tilde{B}^{k k_1}_{k_2 k_3} = \tilde{B}^{k_1 k}_{k_2 k_3} = \tilde{B}^{k k_1}_{k_3 k_2} = -(\tilde{B}^{k_2 k_3}_{k k_1})^*.$$

This transformation is canonical up to the order of $|b_k|^4$.

Classical variables b_k

Here

$$\Gamma^{k}_{k_1 k_2} = -\frac{1}{2}\frac{V^{k}_{k_1 k_2}}{\omega_k - \omega_{k_1} - \omega_{k_2}} \qquad \Gamma_{k k_1 k_2} = -\frac{1}{2}\frac{U_{k k_1 k_2}}{\omega_k + \omega_{k_1} + \omega_{k_2}}.$$

Simplest choice $\tilde{B}^{k k_1}_{k_2 k_3} = 0.$

$$\mathcal{H} = \int \omega_k b_k b^*_k dk + \frac{1}{4}\int T^{k_2 k_3}_{k k_1} b^*_k b^*_{k_1} b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 + \ldots ;$$

Zakharov equation

$$i\dot{b} = \omega_k b_k + \frac{1}{2}\int T_{kk_1}^{k_2 k_3} b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3$$

Zakharov equation

$$i\dot{b} = \omega_k b_k + \frac{1}{2}\int T_{kk_1}^{k_2k_3} b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3$$

$$T_{k_2k_3}^{kk_1} = W_{k_1k}^{k_2k_3} -$$
$$-V_{k_2\,k-k_2}^{k} V_{k_1\,k_3-k_1}^{k_3}\left[\frac{1}{\omega_{k_2}+\omega_{k-k_2}-\omega_k}+\frac{1}{\omega_{k_1}+\omega_{k_3-k_1}-\omega_{k_3}}\right]-$$
$$-V_{k_2\,k_1-k_2}^{k_1} V_{k\,k_3-k}^{k_3}\left[\frac{1}{\omega_{k_2}+\omega_{k_1-k_2}-\omega_{k_1}}+\frac{1}{\omega_{k}+\omega_{k_3-k}-\omega_{k_3}}\right]-$$
$$-V_{k_3\,k-k_3}^{k} V_{k_1\,k_2-k_1}^{k_2}\left[\frac{1}{\omega_{k_3}+\omega_{k-k_3}-\omega_k}+\frac{1}{\omega_{k_1}+\omega_{k_2-k_1}-\omega_{k_2}}\right]-$$
$$-V_{k_3\,k_1-k_3}^{k_1} V_{k\,k_2-k}^{k_2}\left[\frac{1}{\omega_{k_3}+\omega_{k_1-k_3}-\omega_{k_1}}+\frac{1}{\omega_{k}+\omega_{k_2-k}-\omega_{k_2}}\right]-$$
$$-V_{kk_1}^{k+k_1} V_{k_2k_3}^{k_2+k_3}\left[\frac{1}{\omega_{k+k_1}-\omega_k-\omega_{k_1}}+\frac{1}{\omega_{k_2+k_3}-\omega_{k_2}-\omega_{k_3}}\right]-$$
$$-U_{-k-k_1kk_1} U_{-k_2-k_3k_2k_3}\left[\frac{1}{\omega_{k+k_1}+\omega_k+\omega_{k_1}}+\frac{1}{\omega_{k_2+k_3}+\omega_{k_2}+\omega_{k_3}}\right]$$

$T_{kk_1}^{k_2k_3}$ vanishes

On the resonant manifolld

$$k + k_1 = k_2 + k_3, \qquad \omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3},$$

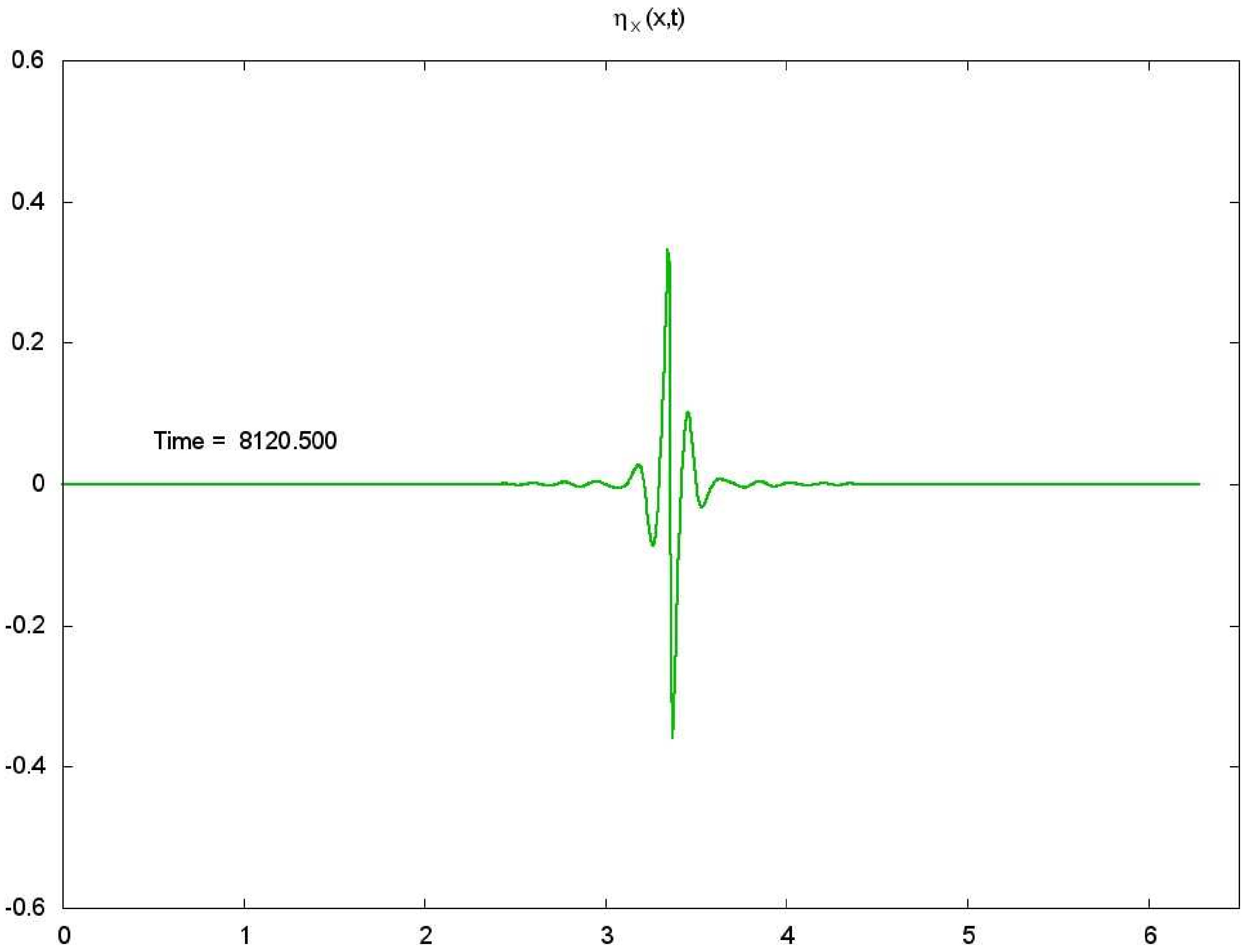
$$\begin{aligned} k &= a(1+\zeta)^2, \\ k_1 &= a(1+\zeta)^2\zeta^2, \\ k_2 &= -a\zeta^2, \\ k_3 &= a(1+\zeta+\zeta^2)^2; \end{aligned}$$

here $0 < \zeta < 1$ and $a > 0$.

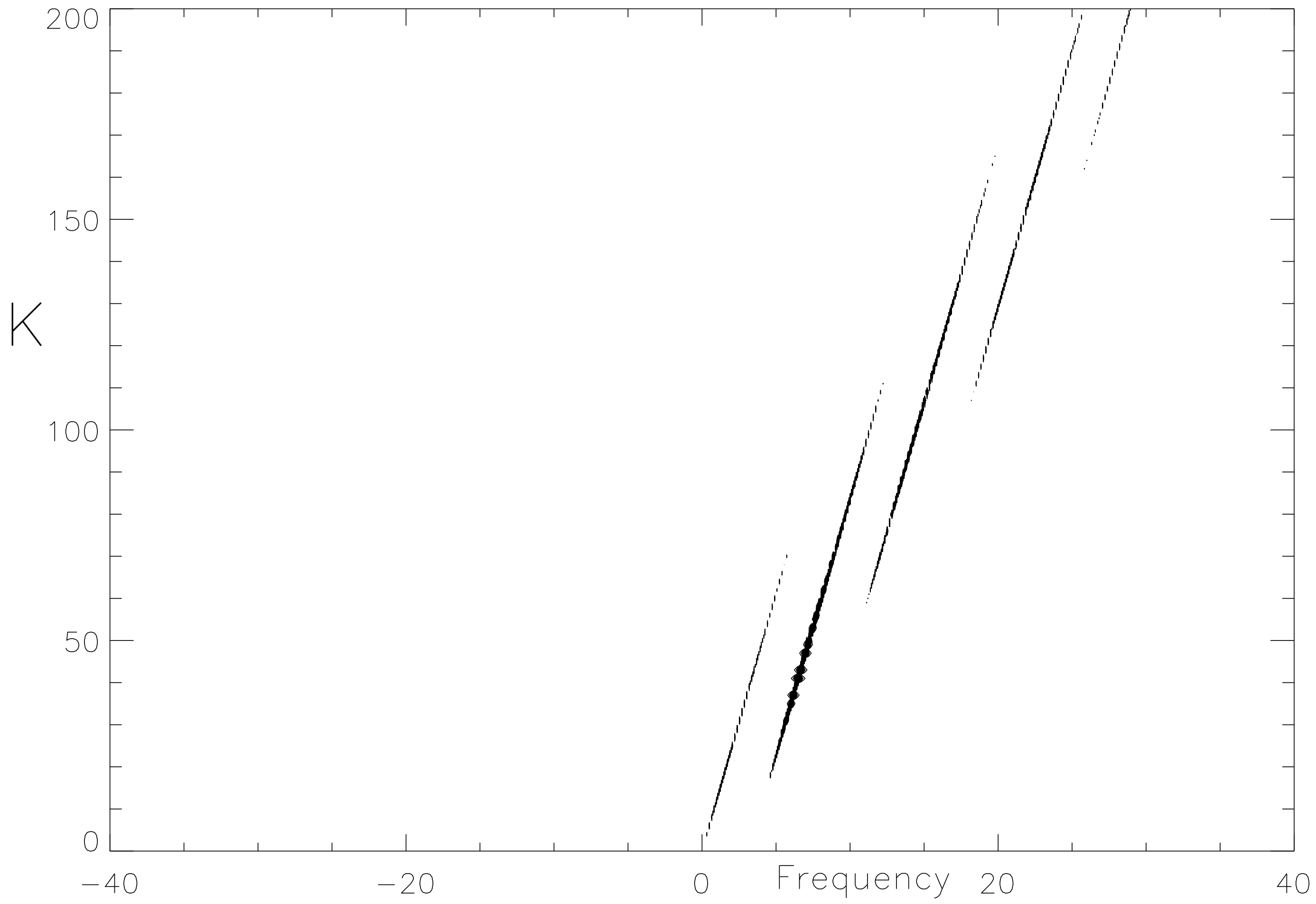
$$T_{kk_1}^{k_2k_3} = 0!$$

If so, it is possible to exclude four-wave interactions ?

Giant breather



k-ω spectrum of giant breather



Fact and observation

- Waves moving in the same direction preserve this property (there is no backward radiation).
  So we can consider waves with $k > 0$.
- Nontrivial 4-wave resonance is absent. So, only wave scattering to itself is important.

$$k + k_1 = k + k_1, \qquad \omega_k + \omega_{k_1} = \omega_k + \omega_{k_1},$$

This scattering is decribed by diagonal part of $T_{kk_1}^{k_2k_3}$, which is

$$T_{kk_1}^{kk_1} = \hat{T}_{kk_1} = \frac{1}{4\pi}|k||k_1|(|k+k_1| - |k-k_1|).$$

Choice of canonical transformation

Using this diagonal part ($\hat{T}_{kk_1}$), one can construct the following function:

$$\tilde{T}^{kk_1}_{k_2k_3} = \left[\frac{1}{2}(T_{kk_2} + T_{kk_3} + T_{k_1k_2} + T_{k_1k_3}) - \frac{1}{4}(T_{kk} + T_{k_1k_1} + T_{k_2k_2} + T_{k_3k_3})\right]\theta(kk_1k_2k_3)$$

$\tilde{T}^{kk_1}_{kk_1}$ coincides with original four-wave coefficient on the resonant manifold. Choose $\tilde{B}^{kk_1}_{k_2k_3}$ as follow

$$\tilde{B}^{kk_1}_{k_2k_3} = \frac{\tilde{T}^{kk_1}_{k_2k_3} - T^{kk_1}_{k_2k_3}}{\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}}$$

$$\mathcal{H} = \int \omega_k b_k b^*_k dk + \frac{1}{4}\int \tilde{T}^{k_2k_3}_{kk_1} b^*_k b^*_{k_1} b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 d$$

Compact Hamiltonian

$$\mathcal{H} = \int b^*(g\hat{K})^{1/2} b dx + \frac{1}{4}\int |b'|^2 \left[\frac{i}{2}(bb'^* - b^*b') - \hat{K}|b|^2\right]$$

Corresponding dynamical equation is

$$i\frac{\partial b}{\partial t} = \hat{(}gK)^{1/2} b \;+\; \frac{i}{8}\left[b^*\frac{\partial}{\partial x}(b'^2) - \frac{\partial}{\partial x}(b^{*\prime}\frac{\partial}{\partial x}b^2)\right] \;-\; \frac{1}{4}\left[b\cdot\hat{K}(|b'|^2) - \frac{\partial}{\partial x}(b'\hat{K}(|b|^2))\right].$$

Monochromatic wave

$$b(x) = B_0 e^{i(k_0 x - \omega_0 t)}, \qquad B_0\text{-is arbitrary}$$

is the simplest solution. One can get the following relation

$$\omega_0 = \omega_{k_0} + \frac{1}{2} k_0^3 |B_0|^2.$$

$$|B_0|^2 = \frac{\omega_{k_0}}{k_0} \eta_0^2,$$

One can recover well known Stokes correction to the frequency due to finite wave amplitude.

$$\omega_0 = \omega_{k_0} (1 + \frac{1}{2} k_0^2 |\eta_0|^2).$$

NLSE, Dysthe and beyond

Equation for amplitude of the wave train can be also easily derived. Let us introduce envelope $B(x,t)$ so that

$$b(x,t) = B(x,t)e^{i(k_0 x - \omega_0 t)}.$$

$B(x,t)$ is also normal hamiltonian variable and Hamiltonian for it is the following:

$$\mathcal{H} = \int B^*(\hat{\omega}_{k_0+k} - \omega_{k_0})B dx +$$
$$+ \frac{1}{4}\int |B' + ik_0 B|^2 \left[\frac{i}{2}(BB'^* - B^*B') + k_0|B|^2 - \hat{K}\right.$$

NLSE, Dysthe and beyond

$B(x,t)$ can be slowly varing function. Then expand $(\hat{\omega}_{k_0+k} - \omega_{k_0})$ and neglect derivatives in nonlinear term one can get NLSE

$$i\dot{B} = -i\omega'_{k_0}B' - \frac{1}{2}\omega''_{k_0}B'' + \frac{1}{2}k_0^3|B|^2B.$$

If we keep first derivatives in the Hamiltonian we derive Dysthe equation.

$$\begin{aligned} \mathcal{H} &= \int B^*(\hat{\omega}_{k_0+k} - \omega_{k_0})Bdx + \\ &+ \frac{1}{4}\int k_0^2|B|^2\left[\frac{3i}{2}(BB'^* - B^*B') + k_0|B|^2 - \hat{K}|B|^2\right] \end{aligned}$$

Modulational instability of monochromatic wave

Consider solution as follow:

$$b = (B_0 + \delta b(x,t))e^{i(k_0 x - \omega_0 t)}$$

where $B_0 = \text{const}$. Linearized equation for $b(x,t)$ has solution as follow

$$\delta b \Rightarrow \delta b e^{\gamma_k t + i(kx - \Omega_k t)},$$

Then for growth rate γ_k the following formula is valid:

$$\gamma_k^2 = \frac{1}{8}\frac{\omega_{k_0}^2}{k_0^4}(1 - \mathbf{6}\mu^{\mathbf{2}})k^2 \left[\mu^2(k_0 - \frac{|\mathbf{k}|}{\mathbf{2}})^2 - \frac{k^2}{8}\right].$$

Going to 3D fluid

$$\mathcal{H} = \int b^* \hat{\omega}_k b dx + \frac{1}{4} \int |b'|^2 \left[\frac{i}{2}(bb'^* - b^* b') - \hat{K}|b|^2 \right] dx.$$

In the spirit of Kadomtsev-Petviashvili equation for Korteveg-de-Vries equation:

$$\mathcal{H} = \int b^* \hat{\omega}_{k_x,k_y} b dx dy + \frac{1}{4} \int |b'_x|^2 \left[\frac{i}{2}(bb'^*_x - b^* b'_x) - \hat{K}_x |b|^2 \right.$$

Spatial equation

Start with dispersion relation:

$$\omega = \sqrt{k} + 2\int T^{kk_1}_{kk_1} |b_{k_1}|^2 dk_1$$

Introduce frequency variable ω instead of k

$$b(k) \to b(\omega), \qquad dk = 2\omega d\omega$$

Approximately

$$k = \omega^2 + 2\int \tilde{T}^{\omega\omega_1}_{\omega\omega_1} |b_{\omega_1}|^2 \omega_1 d\omega_1$$

$$\tilde{T}^{\omega\omega_1}_{\omega\omega_1} = -4\omega\omega_1 T^{\omega\omega_1}_{\omega\omega_1} = -\frac{1}{\pi}\omega^3\omega_1^3(\omega^2 + \omega_1^2 - |\omega^2 - \omega_1^2|)$$

Spatial equation

$$-i\frac{\partial b_\omega}{\partial x} = \omega^2 b_\omega + \frac{1}{2}\int \tilde{T}^{\omega_2\omega_3}_{\omega\omega_1}\delta(\omega+\omega_1-\omega_2-\omega_3)\times$$
$$\times \quad b^*_{\omega_1} b_{\omega_2} b_{\omega_1} d\omega_1 d\omega_2 d\omega_3.$$

$\tilde{T}^{\omega_2\omega_3}_{\omega\omega_1}$ can be recovered from diagonal part

$$\tilde{T}^{\omega_2\omega_3}_{\omega\omega_1} = \tilde{T}^{\omega\omega_2}_{\omega\omega_2} + \tilde{T}^{\omega\omega_3}_{\omega\omega_3} + \tilde{T}^{\omega_1\omega_2}_{\omega_1\omega_2} + \tilde{T}^{\omega_1\omega_3}_{\omega_1\omega_3} -$$
$$- \quad \frac{1}{2}[\tilde{T}^{\omega\omega}_{\omega\omega} + \tilde{T}^{\omega_1\omega_1}_{\omega_1\omega_1} + \tilde{T}^{\omega_2\omega_2}_{\omega_2\omega_2} + \tilde{T}^{\omega_3\omega_3}_{\omega_3\omega_3}]$$