Representation of the Broad-Scale Spectral Form in the Two-Scale Approximation for the Full Boltzmann Integral

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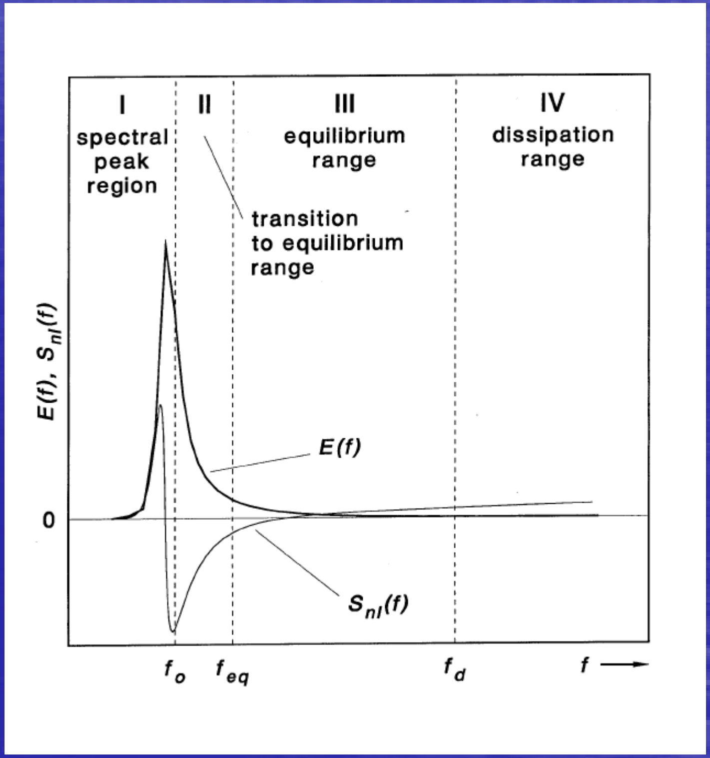
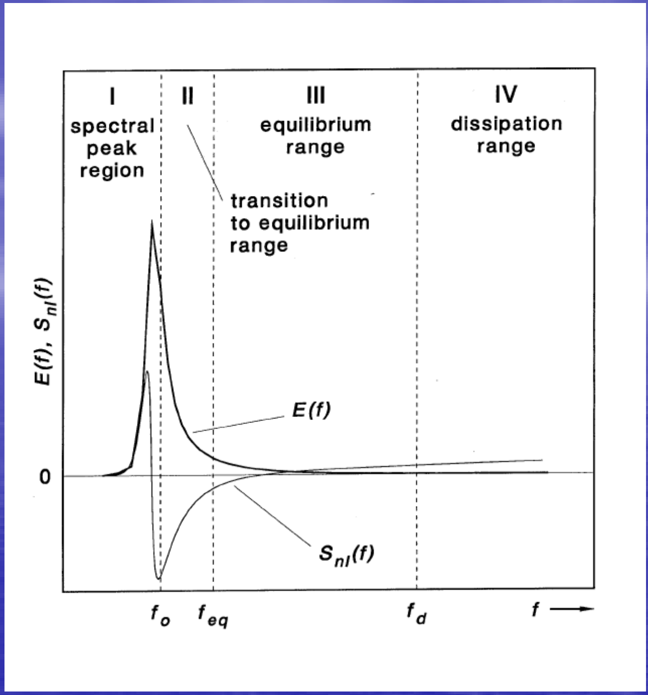
11th International Workshop on Wave Hindcasting and Forecasting
And
2nd Coastal Hazards Symposium

Halifax, N.S., Canada
October 18-23, 2009



MOTIVATION

Nonlinear wave-wave interactions are a primary controlling mechanism in wave generation. The Two-Scale-Approximate (TSA) could help over-come existing problems with the representation of this term; however the transition of this approximation from a theoretical construct to an operational tool is still a "work in progress."



A fundamental question regarding the "operationalization" of the TSA is "how will certain pre-calculated matrices be handled and how much storage will be required to handle them?"

This paper will address these questions.

APPROACH:

- Review the form of the TSA to determine how the formulation is posed in terms of the 2 interacting scales

- Examine the variability of observed directional spectra for the “simple” case of single-peaked wind seas

- Examine the ability of the 2^{nd} scale within the TSA to capture the effects of deviations between the broad-scale spectral form and an actual spectrum

- Examine the effect of superposed swell wave trains upon the total wave-wave interactions

- Evaluate the relative merit of the TSA in terms of a source term for operational models

CONCLUSIONS

- Directional spectra in coastal and offshore areas appear to follow consistent similarity forms of the type first hypothesized by Kitaigorodskii (1962) – even in finite-depth situations.

- For single peaked wind-sea spectra, two parameters appear sufficient to provide a "good" representation of the nonlinear interactions, spectral peakedness and relative depth.

- Storage requirements for pre-calculated terms and coefficients appear to be very manageable.

- The relaxation of a spectral perturbation estimated by the TSA is very close to that of the FBI

- Since wave generation in many situations follows a similarity basis (Badulin et al, 2008), the TSA offers an very good basis for a highly accurate representation of "typical" wave generation cases.

- Interactions with swell could be added for a small range of significance

RECENT PROGRESS IN TSA

Timing studies – TSA faster than DIA

TSA moved from FBI to separate code

- transfer matrices developed
- scaling (finally) complete
- Initial 2 papers published JPO
    - Theory and hypothetical spectra
    - Real spectra
- next manuscript on spectral evolution (here)

Basis for Two-Scale Approximation: Decompose spectrum into 2 parts – a Broad-Scale and a Perturbation Scale

$$
\begin{aligned}
N^3 = {} & \hat{n}_1\hat{n}_3(\hat{n}_4-\hat{n}_2)+\hat{n}_2\hat{n}_4(\hat{n}_3-\hat{n}_1)+ \\
& n_1'n_3'(n_4'-n_2')+n_2'n_4'(n_3'-n_1')+ \\
& \hat{n}_1\hat{n}_3(n_4'-n_2')+\hat{n}_2\hat{n}_4(n_3'-n_1')+ \\
& n_1'n_3'(\hat{n}_4-\hat{n}_2)+n_2'n_4'(\hat{n}_3-\hat{n}_1)+ \\
& \hat{n}_1n_3'(\hat{n}_4-\hat{n}_2)+\hat{n}_2n_4'(\hat{n}_3-\hat{n}_1)+ \\
& n_1'\hat{n}_3(\hat{n}_4-\hat{n}_2)+n_2'\hat{n}_4(\hat{n}_3-\hat{n}_1)+ \\
& \hat{n}_1n_3'(n_4'-n_2')+\hat{n}_2n_4'(n_3'-n_1')+ \\
& n_1'\hat{n}_3(n_4'-n_2')+n_2'\hat{n}_4(n_3'-n_1')
\end{aligned}
$$

$$n = \hat{n} + n'$$

$$S_{nl}(f,\theta) = B + L + X$$

Line 1 contains interactions for only B
Line 2 contains interactions for only L
Lines 3-8 contain cross-interactions between B and L

This approximation to the full integral would be exact if all terms were retained.

$$\frac{\partial n_1}{\partial t} = \iint\int N_A^3 C \left| \frac{\partial W}{\partial n} \right|^{-1} ds\ k_3 d\theta_3 dk_3$$

Scale 1 Terms

$$+$$

$$\iint\int N_B^3 C \left| \frac{\partial W}{\partial n} \right|^{-1} ds\ k_3 d\theta_3 dk_3$$

Retained Terms from L+X

$$+$$

$$\iint\int N_C^3 C \left| \frac{\partial W}{\partial n} \right|^{-1} ds\ k_3 d\theta_3 dk_3$$

Neglected Terms from L+X

The fundamental idea here is to capture the broad-scale distribution of energy parametrically and to allow "local" differences to be treated as shown below. Terms that are neglected tend to contribute in a +/- sense around locus – "s."

This could be a DIA form or a diffusion operator, but we would lose considerable accuracy.

neglected n_2' and n_4' terms

$$\frac{\partial n_1}{\partial t} = B + \iint N_*^3 C \left| \frac{\partial W}{\partial n} \right|^{-1} ds\, k_3 d\theta_3 dk_3 + \ldots\ldots$$

N_*^3 terms neglect terms containing n_2' and n_4' - retain $\hat{n}_2$ and $\hat{n}_4$

$$N_*^3 = \hat{n}_2\hat{n}_4(n_3' - n_1') + n_1'n_3'(\hat{n}_4 - \hat{n}_2) + \hat{n}_1 n_3'(\hat{n}_4 - \hat{n}_2) + n_1'\hat{n}_3(\hat{n}_4 - \hat{n}_2)$$

Note: X is typically 2-3 times larger than L or B. This is why linear sums (neural networks, EOF's, etc.) do not work well for Snl estimation.

$$\frac{\partial n_1}{\partial t} = B\left(\frac{k}{k_0}\right)^{19/2} \left\langle \left(\frac{\beta}{\beta_0}\right) \iint (\hat{n}_1 n_3' + n_1' \hat{n}_3 + n_1' n_3') \Lambda(\hat{n}_2 - \hat{n}_4, \underline{k}_1, k_*, \theta_*, \mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n})\, k_* d\theta_* d\, k_* + \left(\frac{\beta}{\beta_0}\right)^2 \iint (n_1' - n_3') \Lambda(\hat{n}_2 \hat{n}_4, \underline{k}_1, k_*, \theta_*, \mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n})\, k_* d\theta_* dk_* \right\rangle$$

Pre-calculated terms

where

$$\Lambda(\hat{n}_2 - \hat{n}_4, \underline{k}_1, k_*, \theta_*, \mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}) = \oint C \left| \frac{\partial W}{\partial n} \right|^{-1} (\hat{n}_4 - \hat{n}_2) ds$$

$$\Lambda(\hat{n}_2 \hat{n}_4, \underline{k}_1, k_*, \theta_*, \mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}) = \oint C \left| \frac{\partial W}{\partial n} \right|^{-1} \hat{n}_2 \hat{n}_4 d\, s$$

$$\frac{\partial n_1}{\partial t} = \sum_{\delta f_i} \sum_{\delta \theta_j} \mu_{ij\vec{s}p} N_{*_P}^3 + \mu_{ij\vec{s}d} N_{*_D}^3$$

Pumping Diffusion

Note: **This has no free parameters. All coefficients are determined from the full integral.** Still conserves action, energy, and momentum since each elemental transfer is conservative.

JONSWSAP (f^{-5} spectrum)

$$E(f) = \frac{\alpha g^2}{(2\pi)^4} f^{-5} \exp\left[-1.25\left(\frac{f}{f_p}\right)^4\right]\gamma_5^{\Theta_5}$$

where $E(f)$ is the spectral energy density at f,

$$\Theta_5 = \exp\left[\frac{-(f - f_p)^2}{2\sigma^2 f_p^2}\right]$$

and

$\sigma = \sigma_a$ for $f < f_p$

$= \sigma_b$ for $f \le f_p$

Modified f^{-4} spectrum after Resio and Perrie (1989)

$$E(f) = \frac{2\beta g^{3/2}}{(2\pi)^3} f^{-4}\left[z_4\left(\frac{f}{f_p}\right)^4 \exp(-\Theta_4) + 1\right]$$

where

β is the equilibrium range constant as defined in Resio et al. (2004)

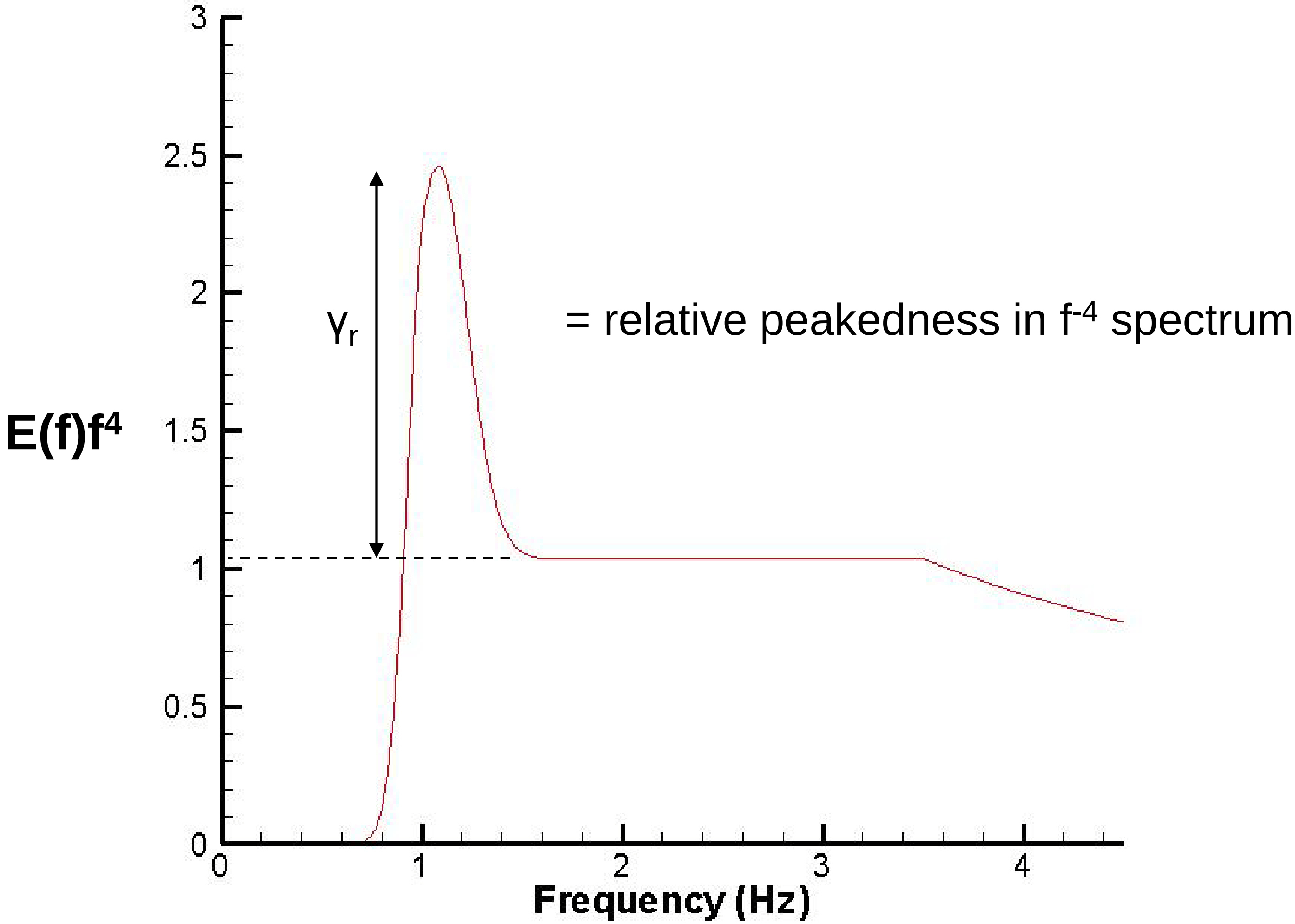
z_4 is a constant = γ_r for $f \le f_p$; $\gamma_r - 1$ for $f > f_p$

γ_r is the relative peakedness as defined in Long and Resio (2007)

Θ_4 is a peakedness factor given by

$$\Theta_4 = \left[\frac{(f - f_p)}{2\sigma f_p}\right]^2$$

with values the same as for the JONSWAP spectrum.

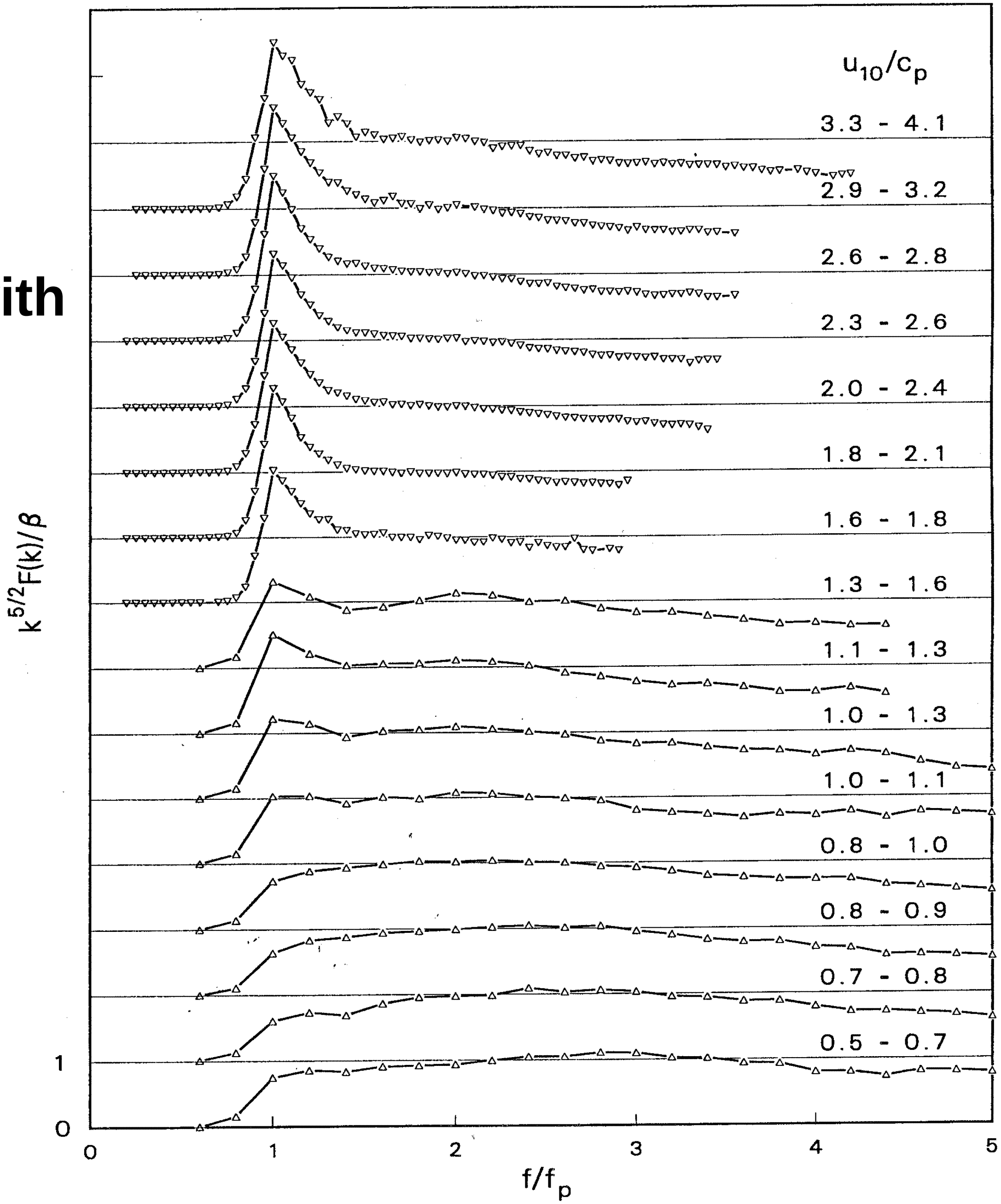


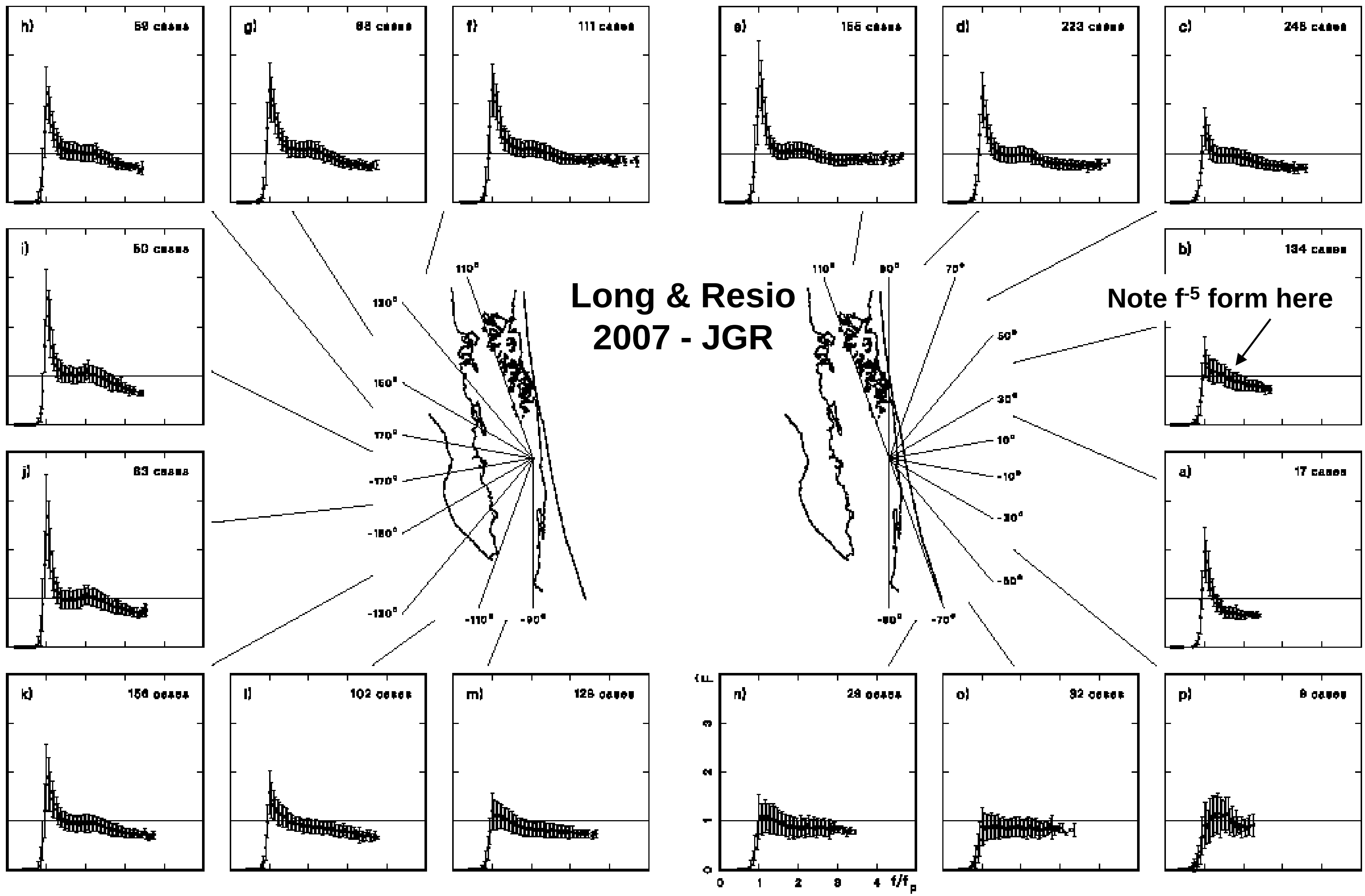
Idealized form for f^{-4} spectrum

Spectra from around the world have shown a pronounced f^{-4} (k^{-3}) form with a transition to f^{-5} form at high frequencies.

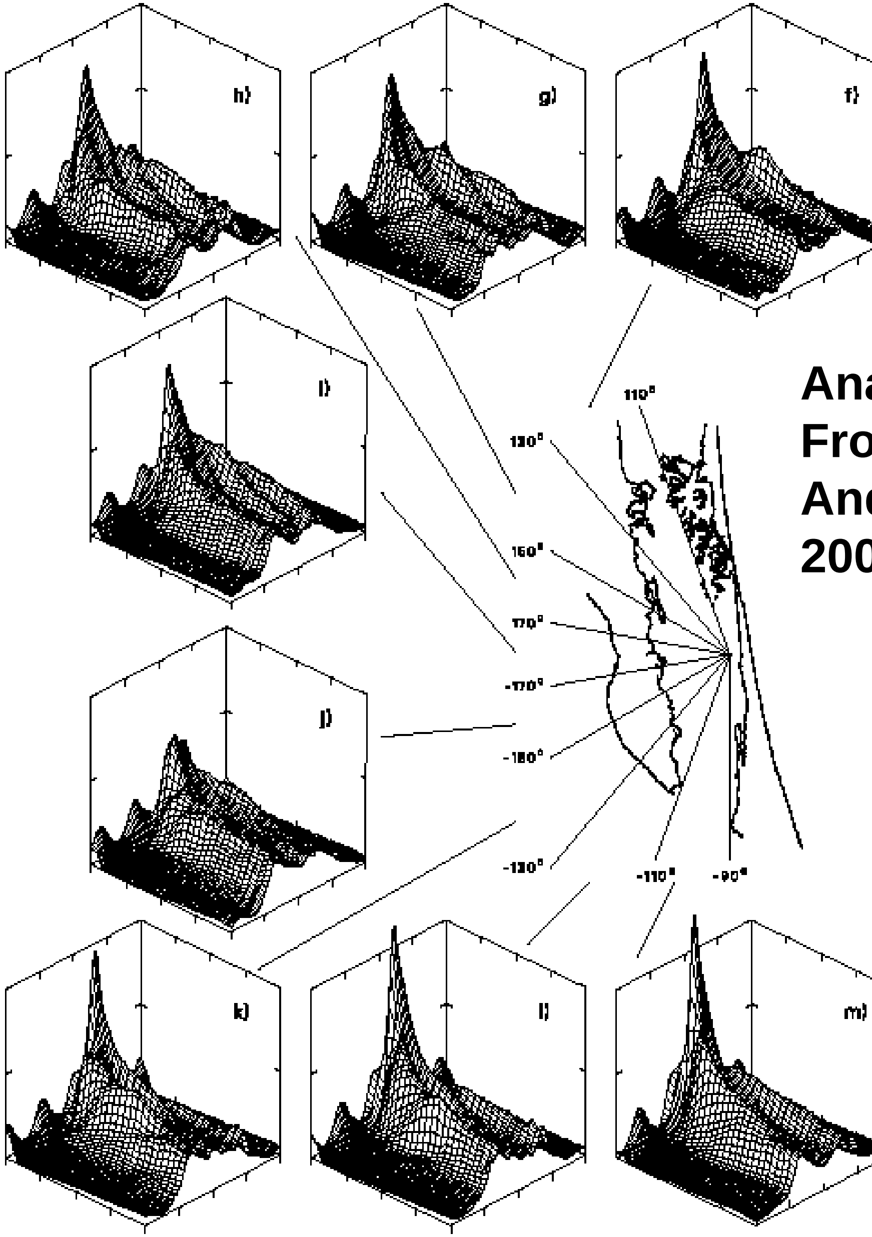
$$E_c(f) \sim E(f)\, f^{4}$$

$$\rightarrow F_c(k) \sim F(k)k^{-5/2}$$

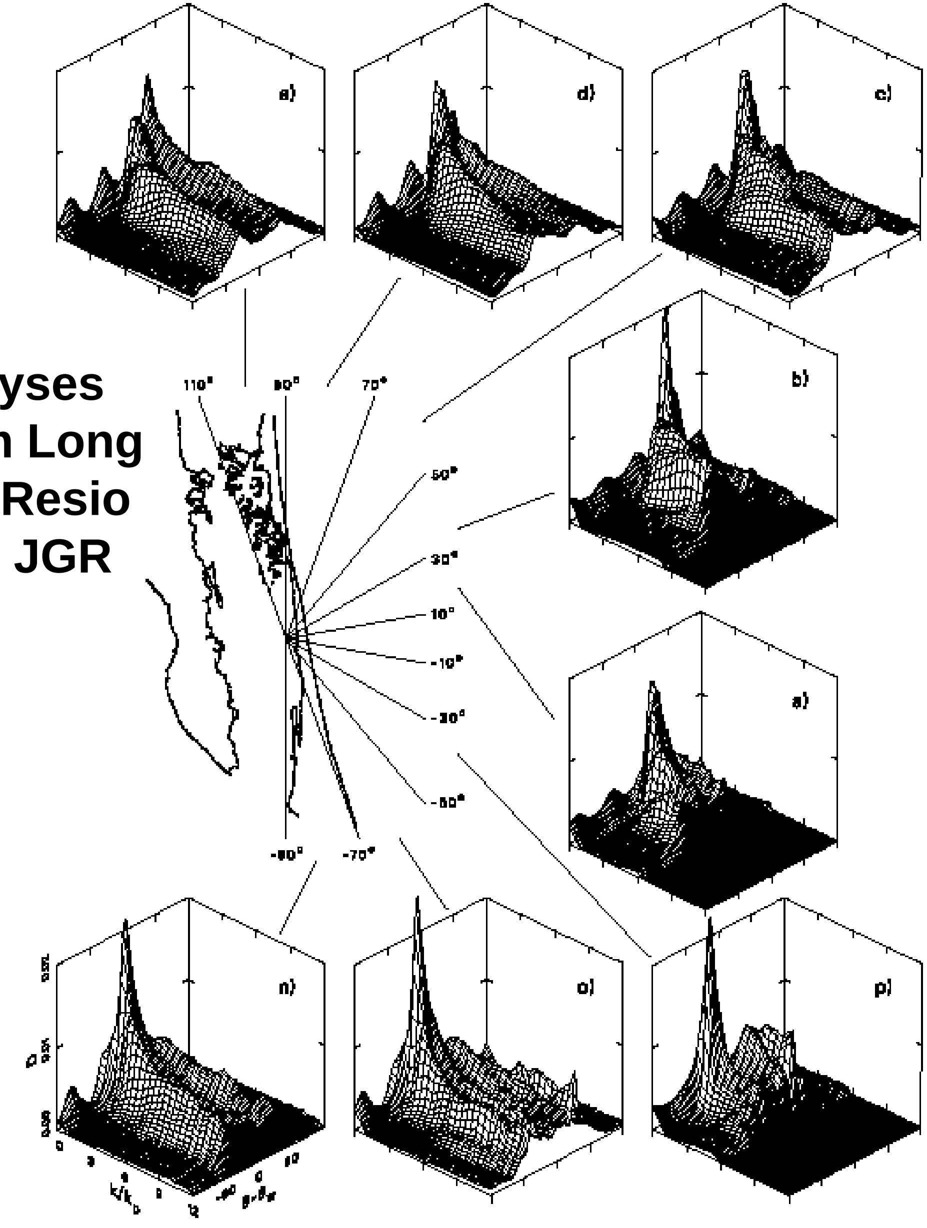


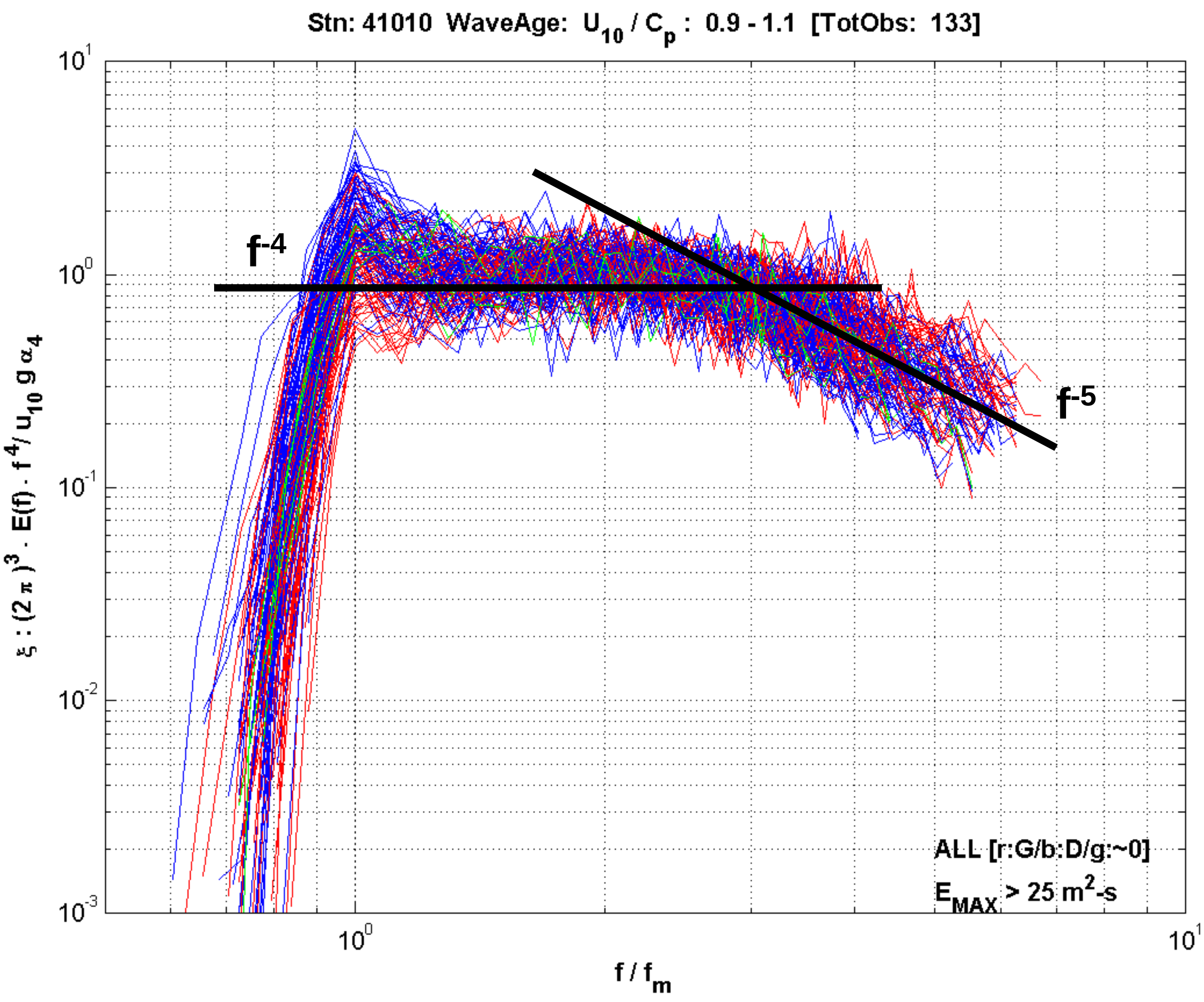


Long & Resio
2007 - JGR
Note f-5 form here
h) 59 cases
g) 68 cases
f) 111 cases
e) 156 cases
d) 223 cases
c) 246 cases
i) 50 cases
b) 134 cases
j) 63 cases
a) 17 cases
k) 156 cases
l) 102 cases
m) 129 cases
n) 29 cases
o) 32 cases
p) 9 cases
f/f_p



Analyses From Long And Resio 2007 JGR





Compensated Spectra from Hurricane Ivan

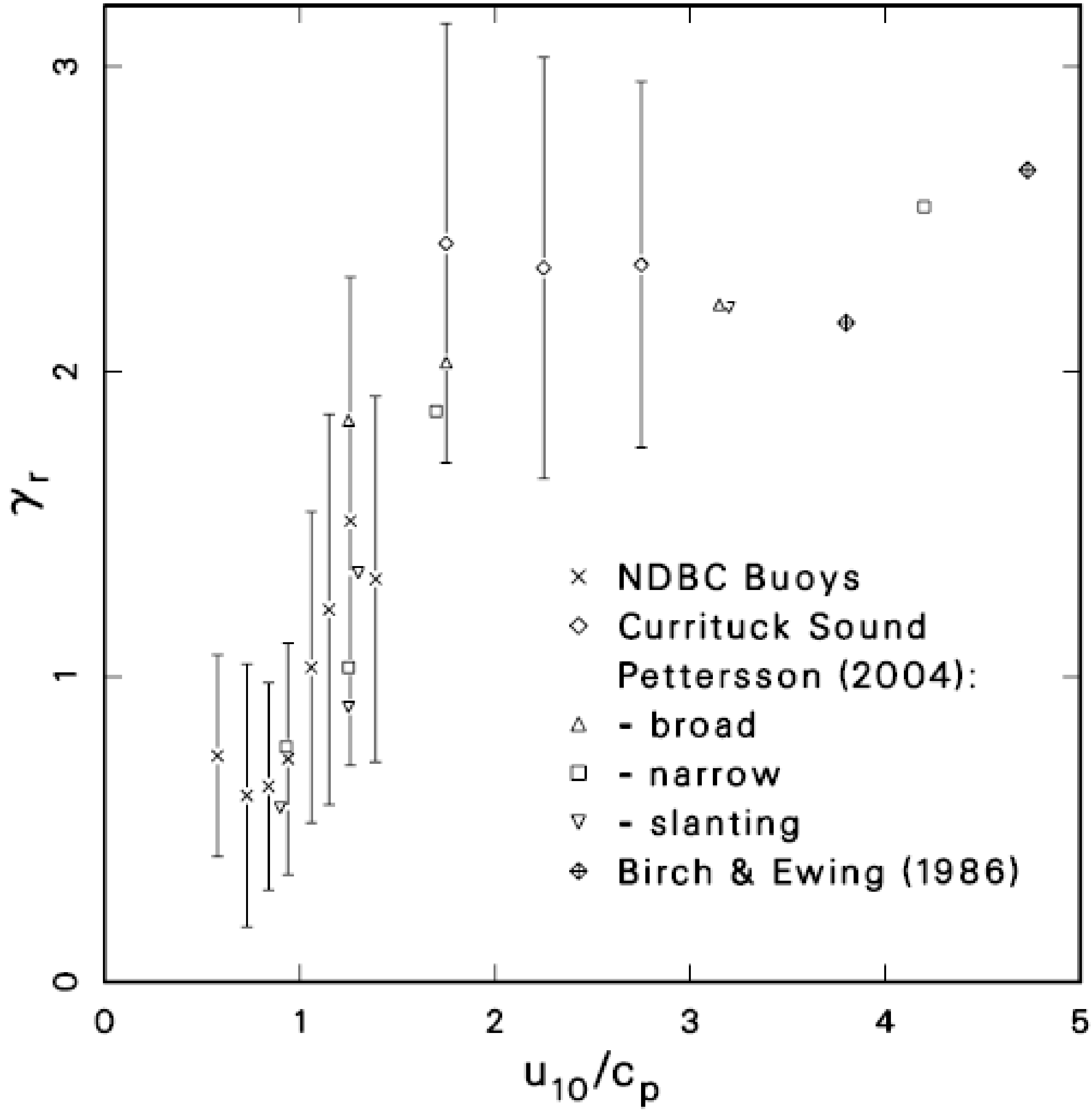
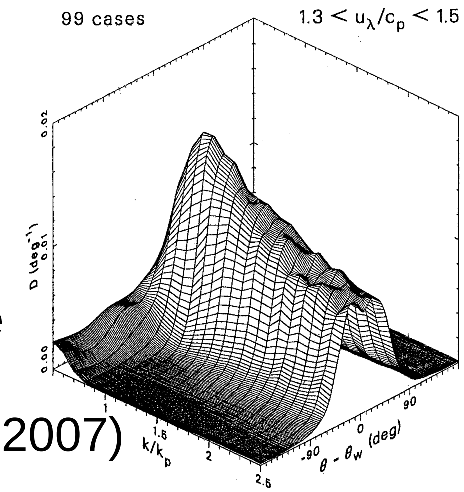


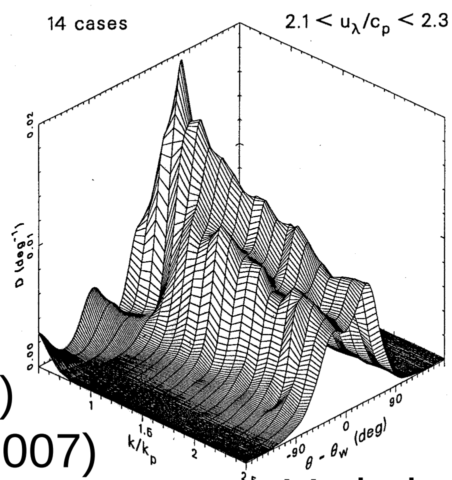
Figure 2. Observed relationship between spectral peakedness and inverse wave age (Long and Resio, 2007). Relationship is not as chaotic as JONSWAP data indicated.

Characteristics of directional distributions of energy:

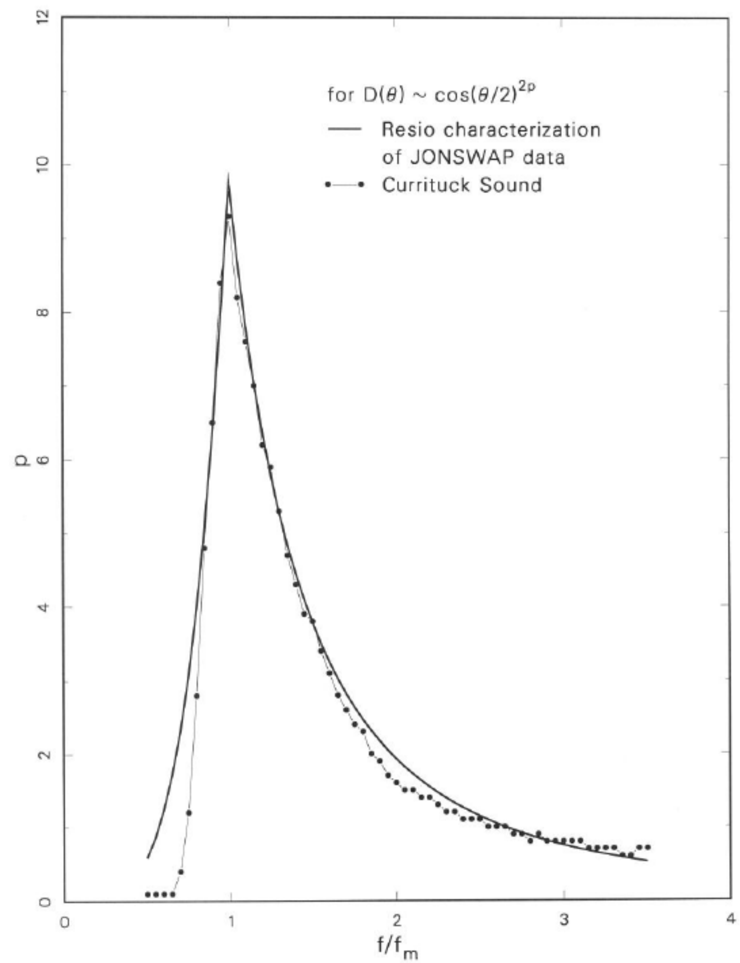
1. **"Young" waves are very bimodal**
2. **"Old" waves approach unimodal**
3. **Both distributions are similar to Hasselmann et al. (1980)**



Low inverse wave age (old waves) (Long and Resio, 2007)

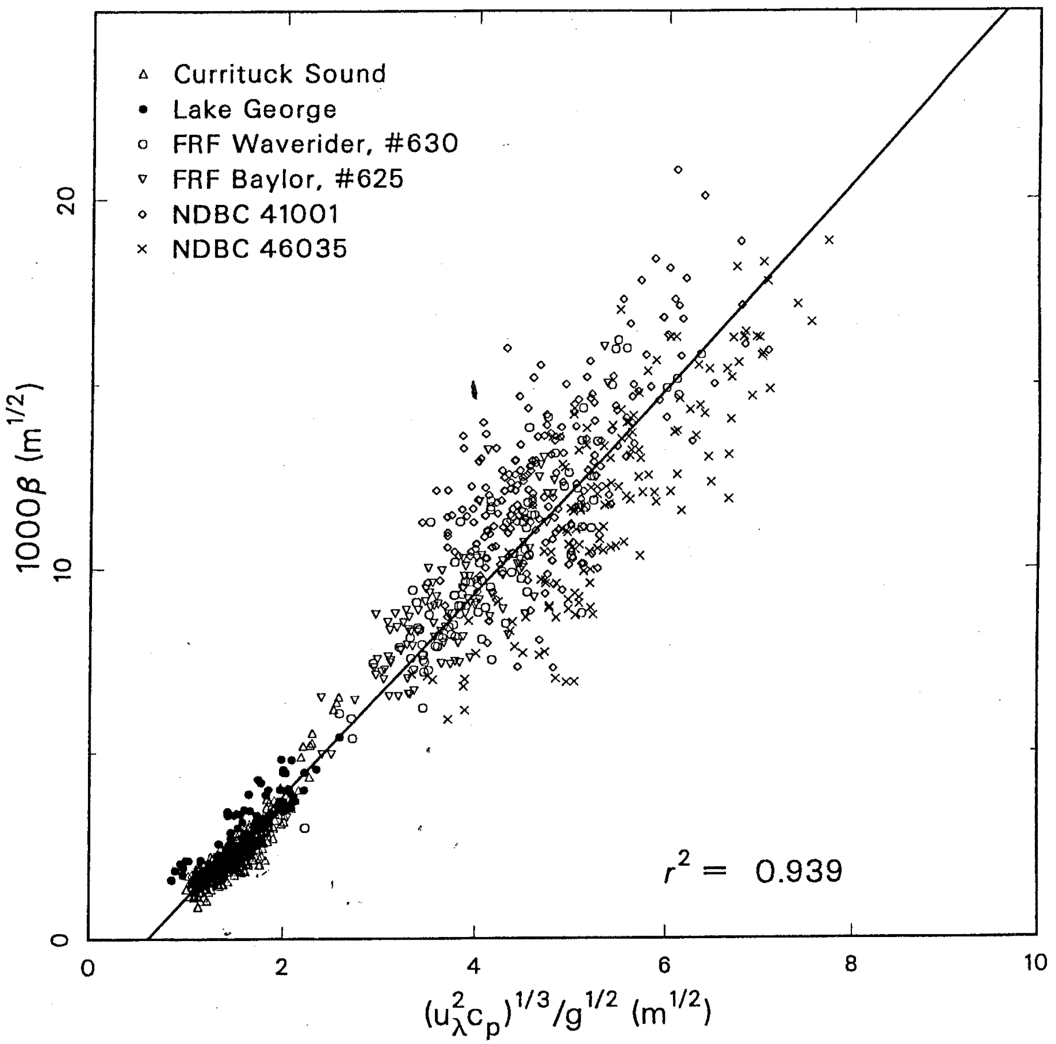


High inverse wave age (young waves) (Long and Resio, 2007)

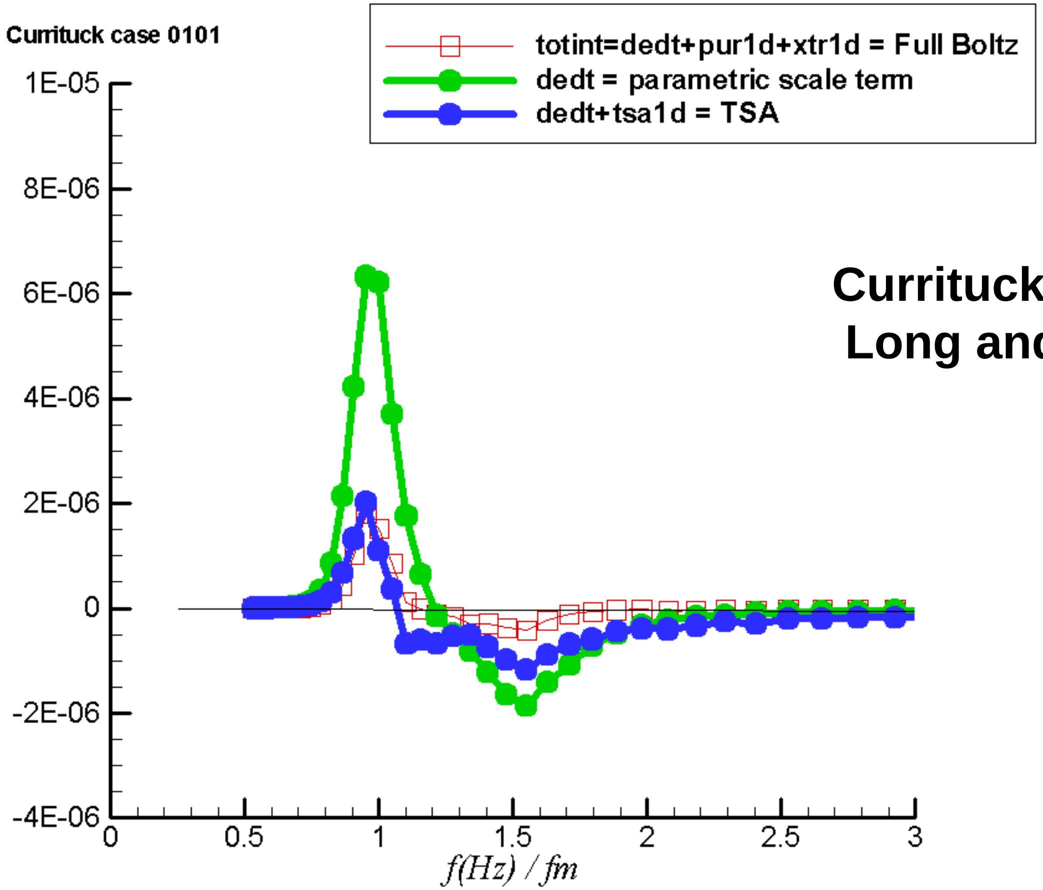


Variation in "n" obtained when fitting a $\cos^{2n}$ function compared to the data from Hasselmann *et al* (1980)

Both f_p and β scale out of the TSA representation.
β scales very well across many sets of experimental data.



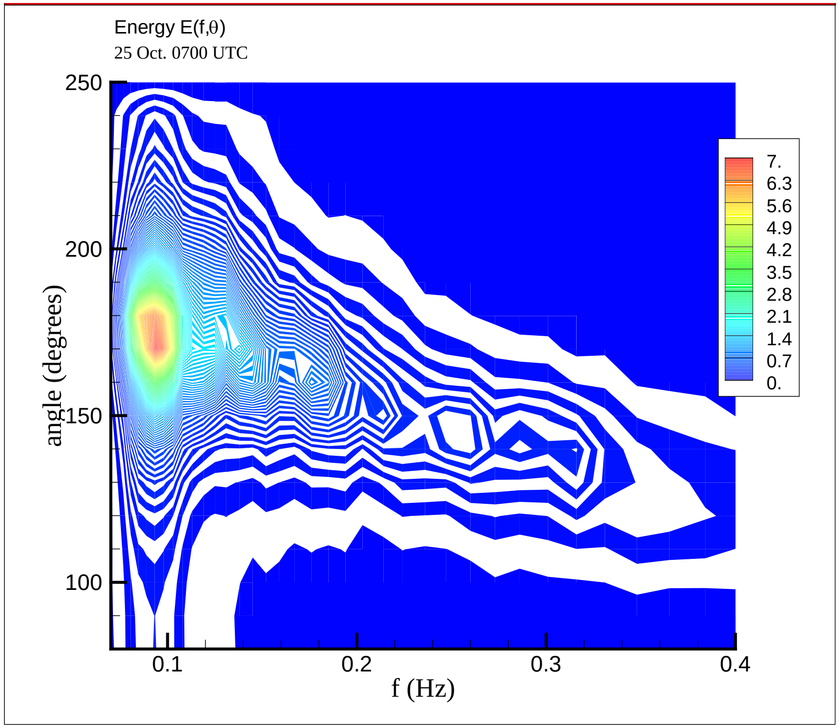
Effectiveness of 2nd term on improving the fit of full TSA over parametric term alone for case of observed bimodal directional distribution to $\cos^6$ distribution.



Currituck Sound case from Long and Resio (2007).

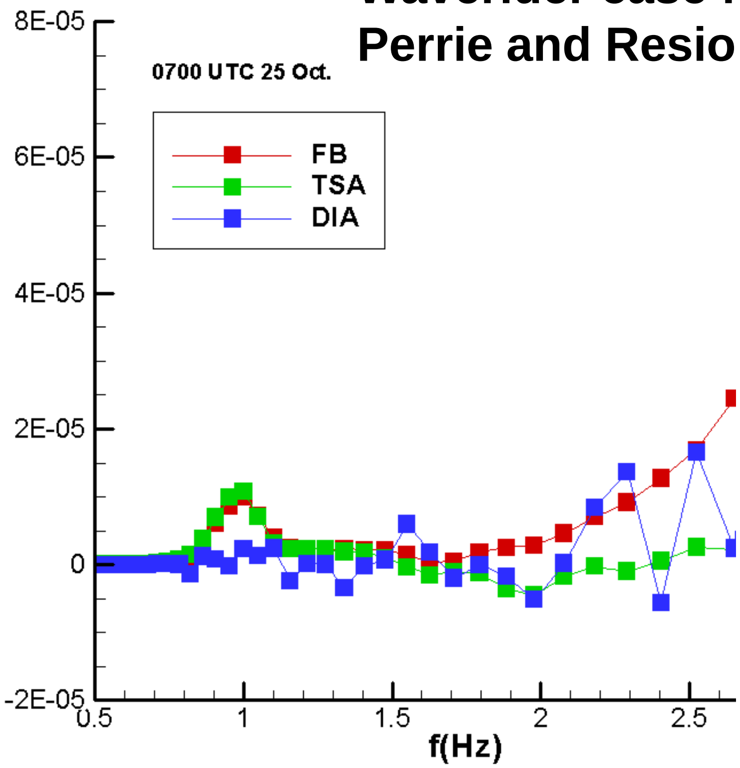
Comparison of parametric term alone in TSA (green line) versus parametric plus second term in the TSA (blue line) to the full integral solution (red line).

Effectiveness of TSA for case of skewed directional distribution to cos^6 distribution.



Contours of directional spectral energies from depth of 17 m offshore of Duck, NC

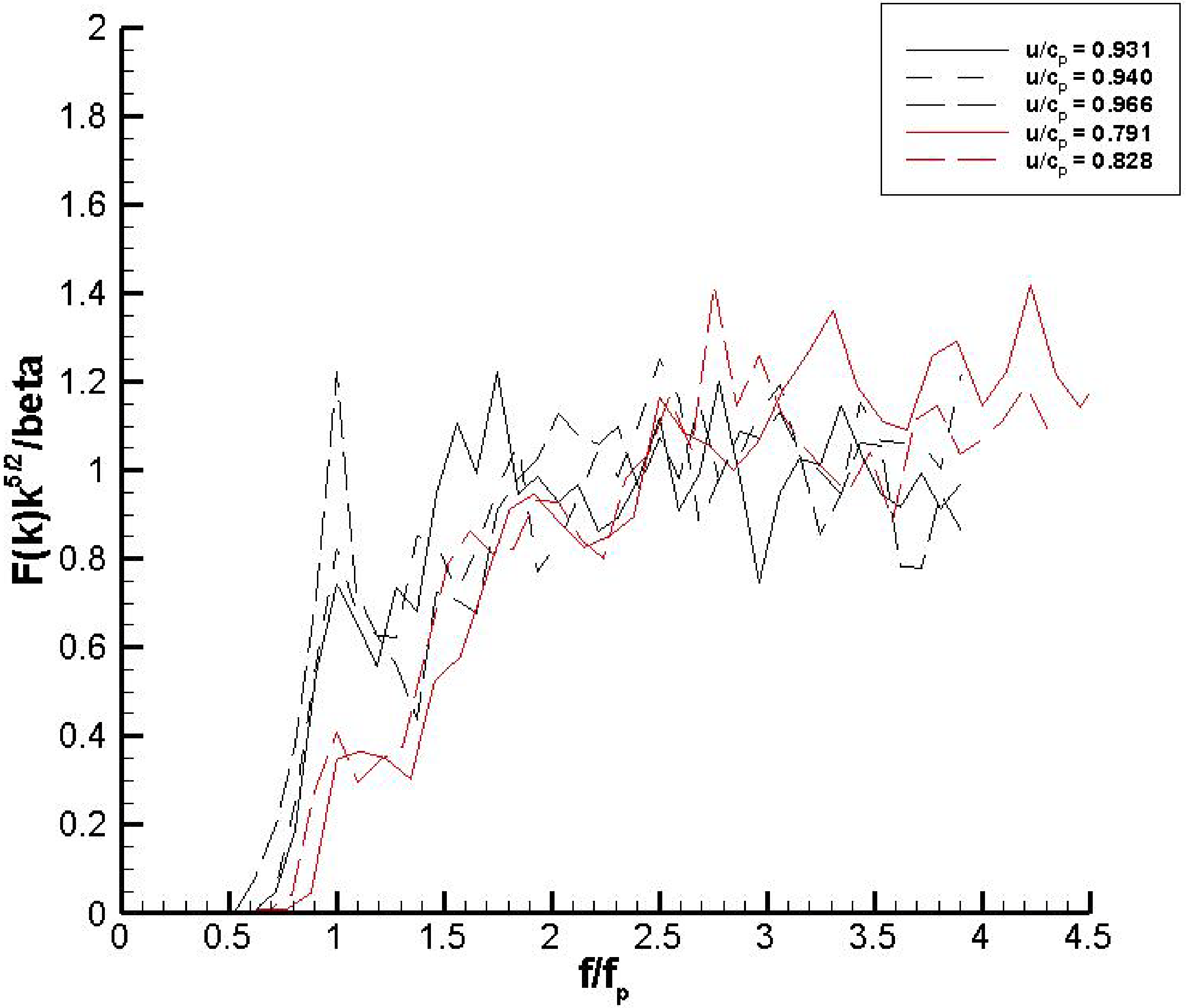
Waverider case from Perrie and Resio (2007)



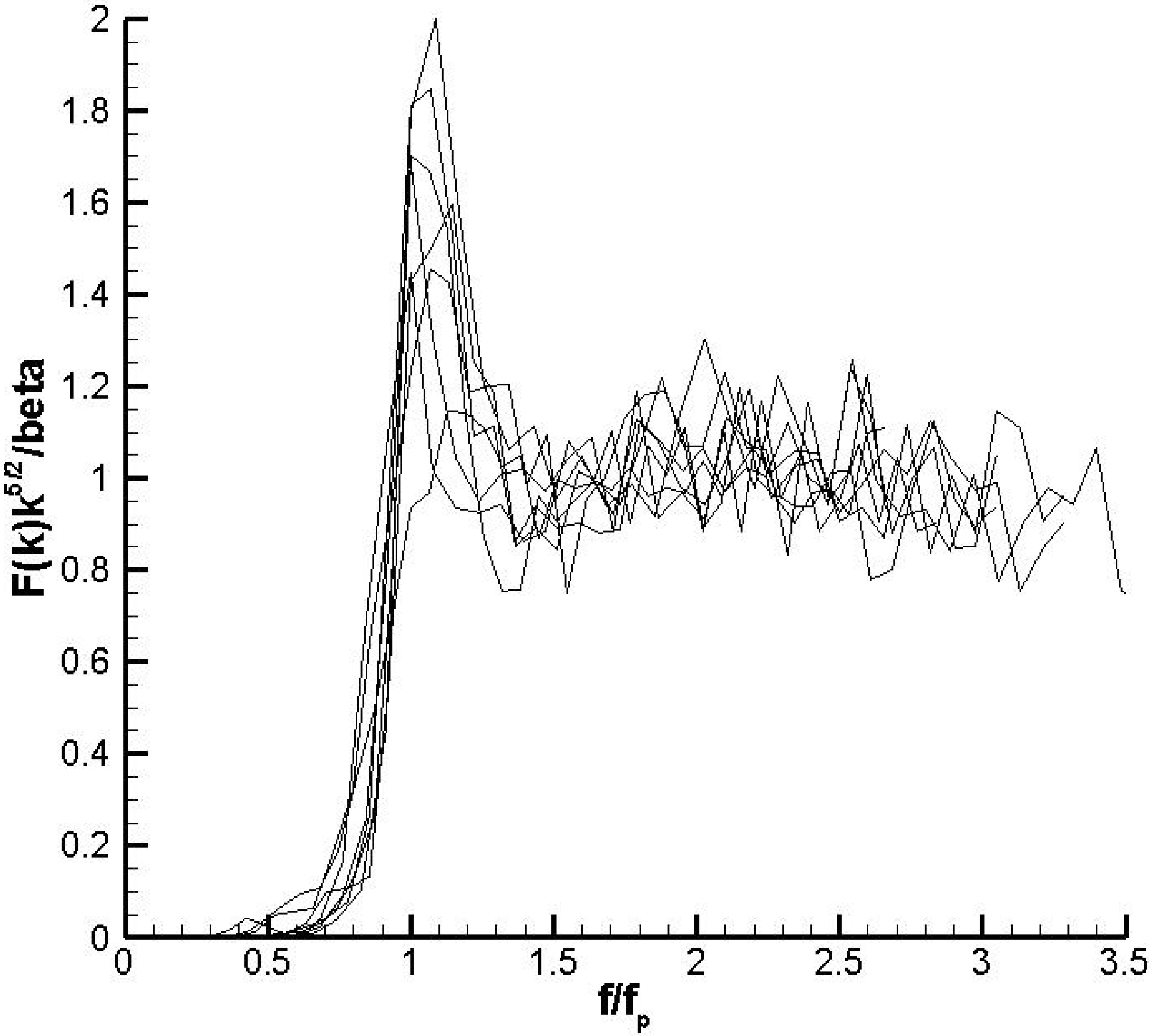
Comparison of TSA solution and DIA for skewed directional distribution.

Some Additional Views: Compensated Spectra

$$E_c(f) \sim E(f) f^4 \rightarrow F_c(k) \sim F(k) k^{-5/2}$$

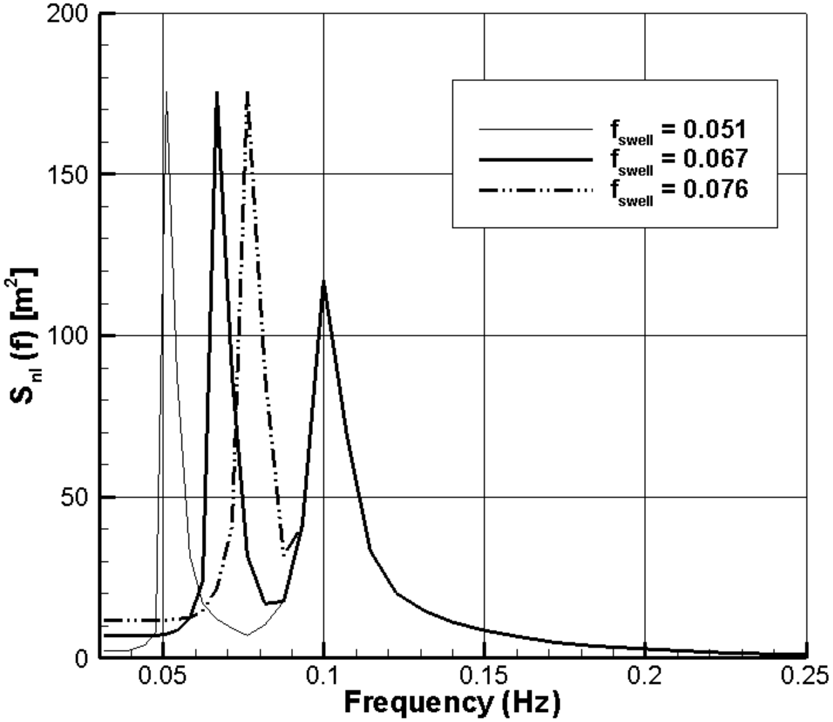


Compensated low-peakedness spectra from waverider in 17 meter depth
off of Duck, NC



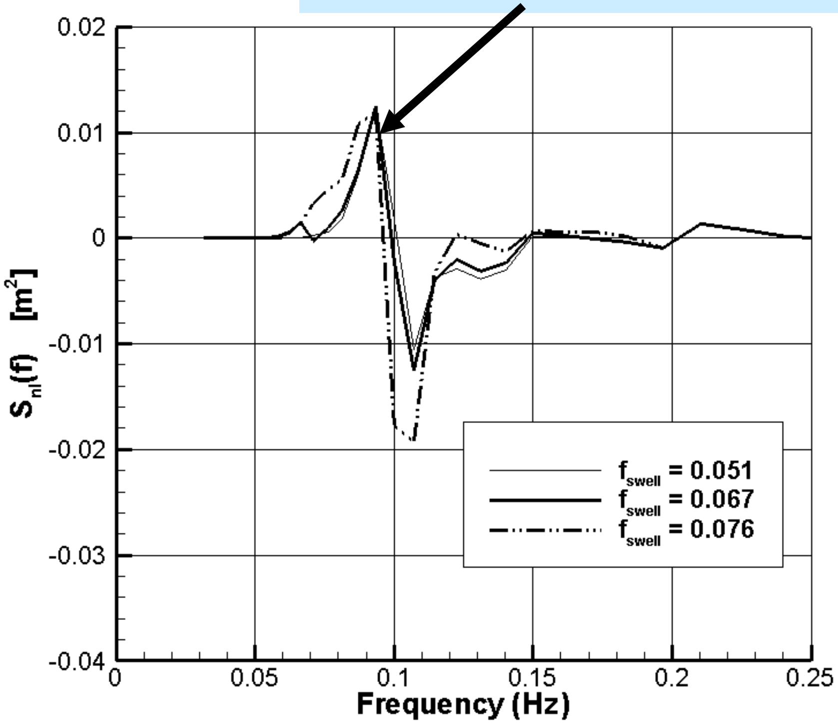
Compensated high-peakedness spectra from waverider in 17 meter depth off of Duck, NC

What about Mixed Sea and Swell??

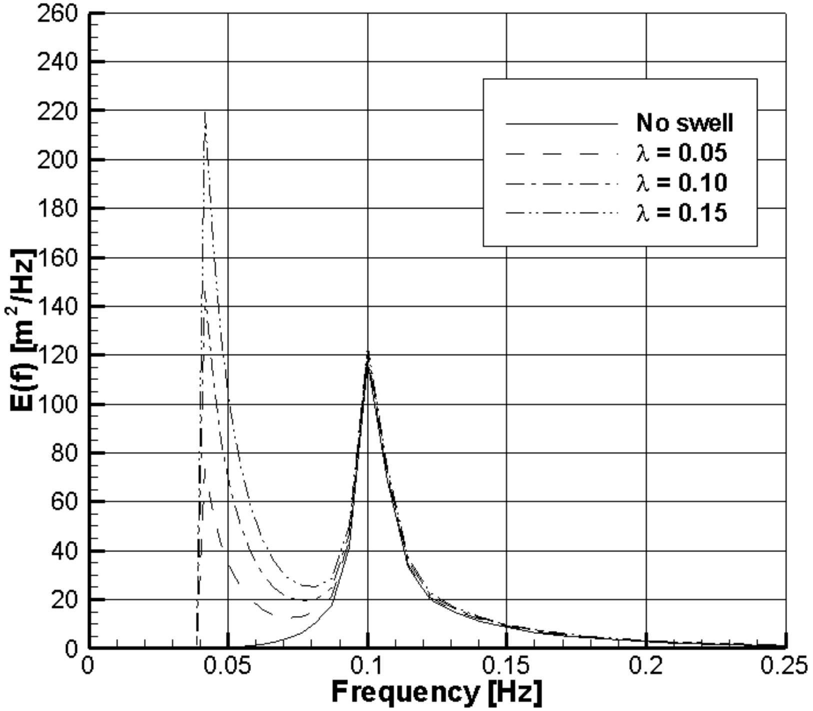


Sea spectrum with 3 swell spectra having different separations

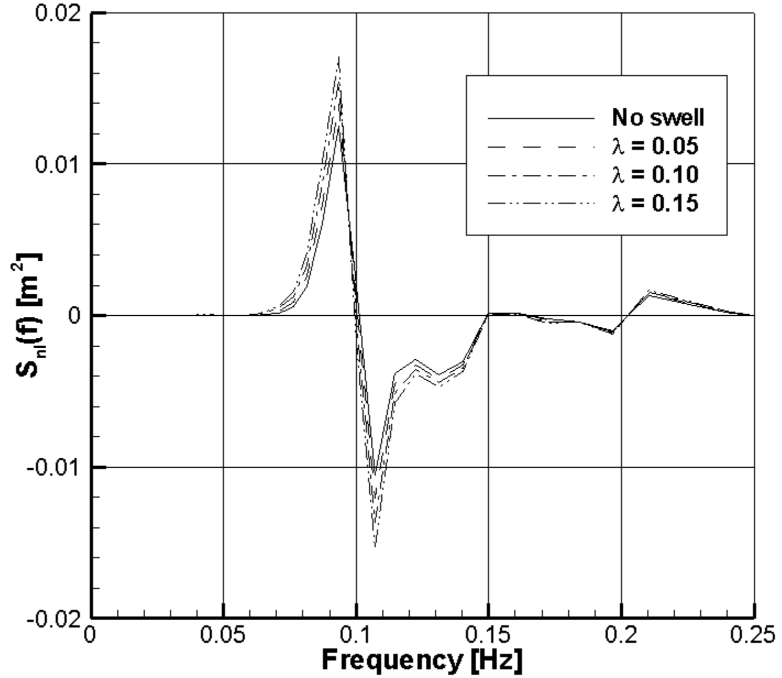
Two cases plot essentially over the no swell line



SNL for sea spectrum with 3 swell spectra having different separations

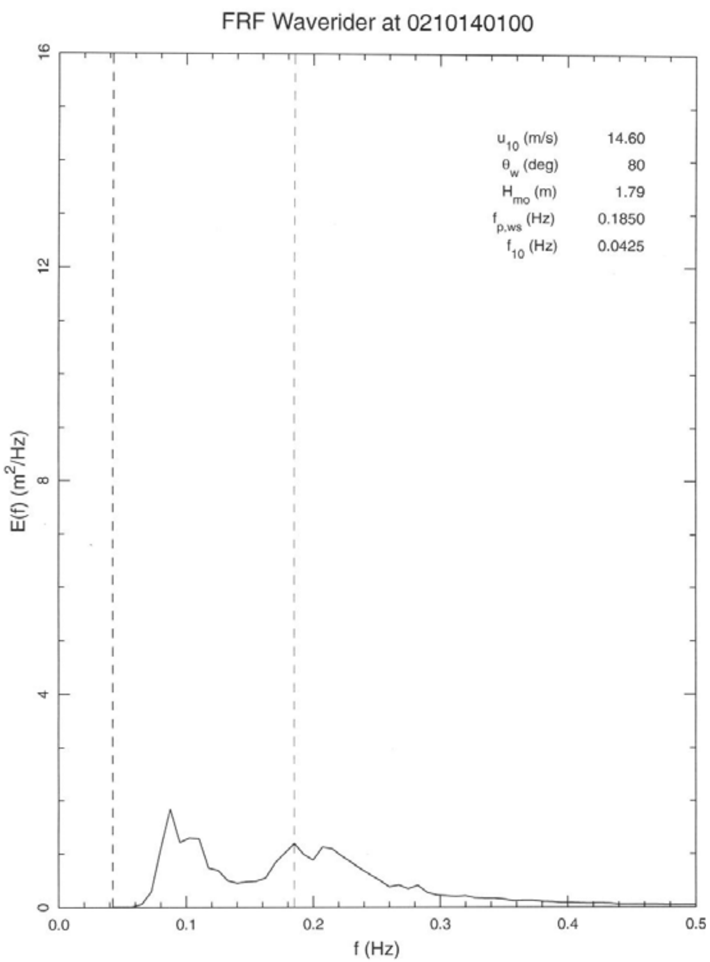


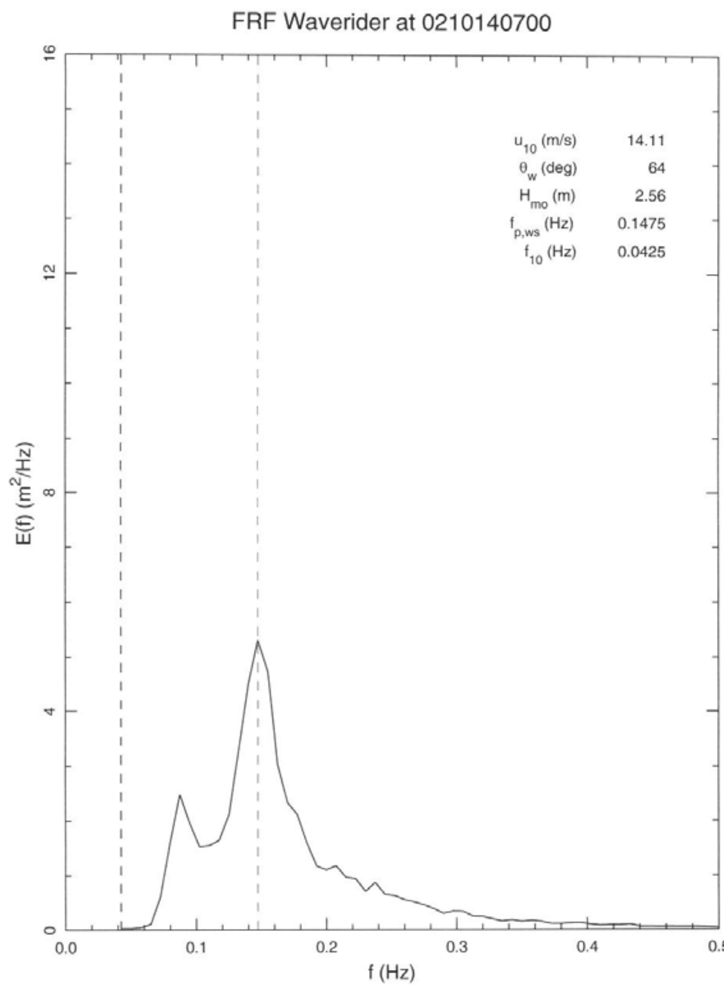
Varying swell steepness

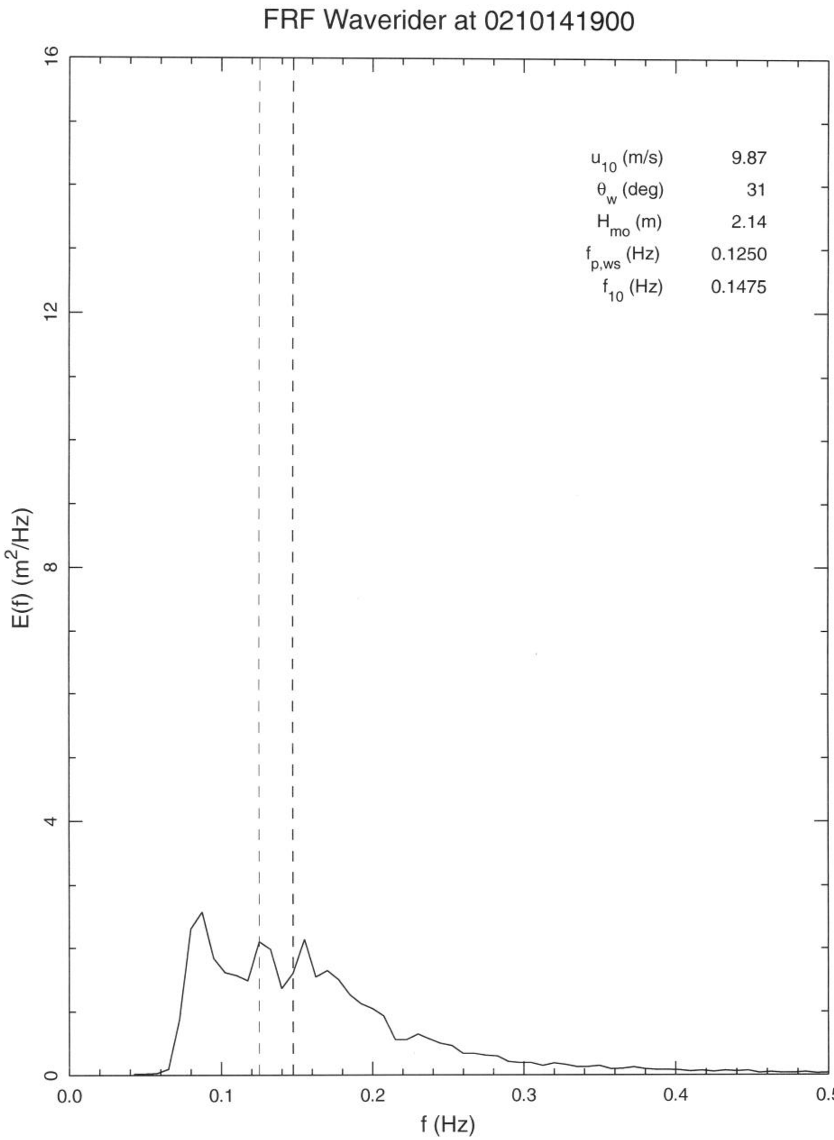


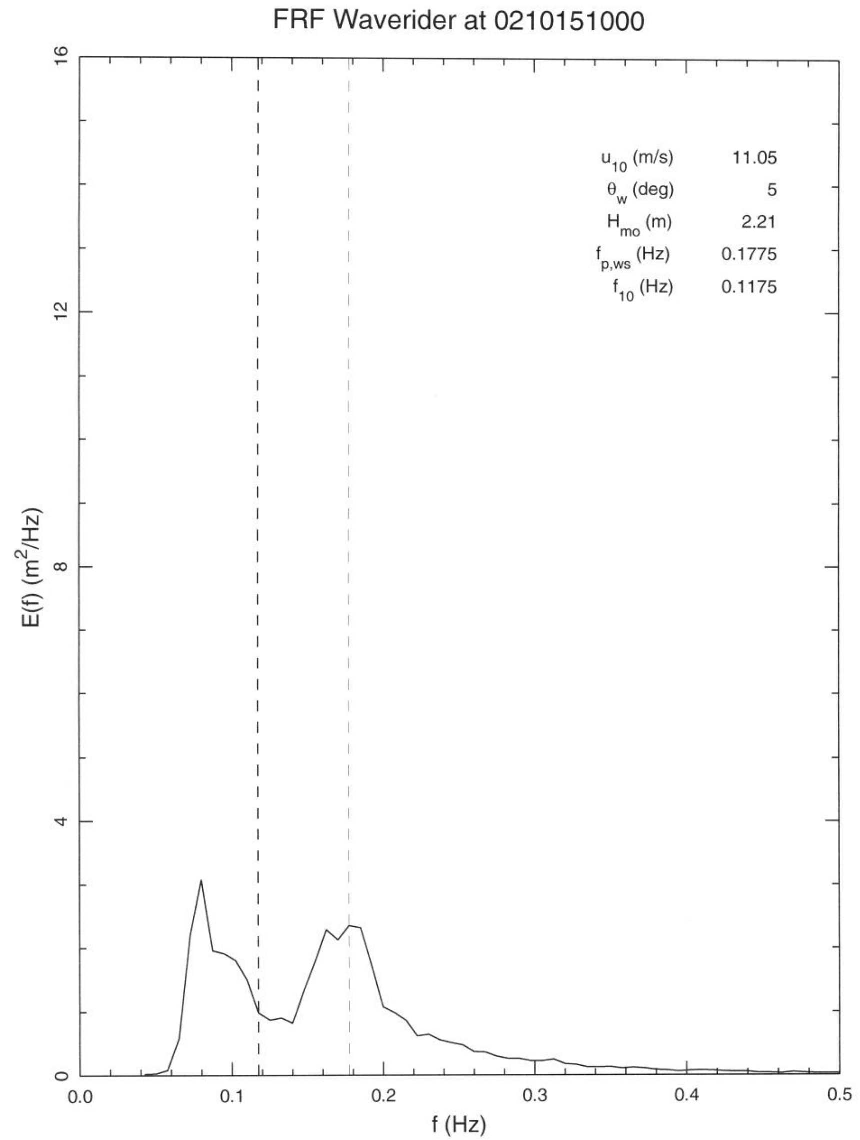
SNL for varying swell steepness

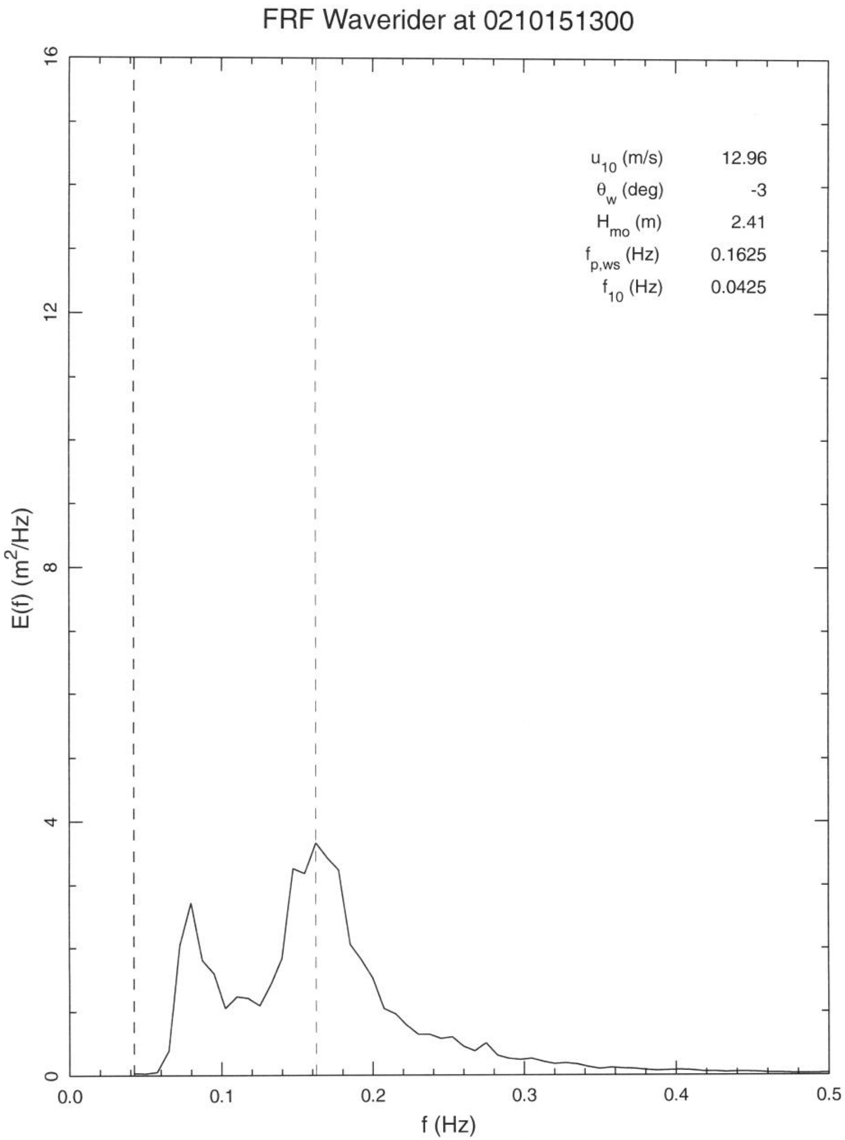
Some Additional Views: Coastal Mixed Sea and Swell Directionally Aligned


FRF Waverider at 0210140100
u_{10} (m/s) 14.60
θ_w (deg) 80
H_{mo} (m) 1.79
$f_{p,ws}$ (Hz) 0.1850
f_{10} (Hz) 0.0425
E(f) (m^2/Hz)
f (Hz)

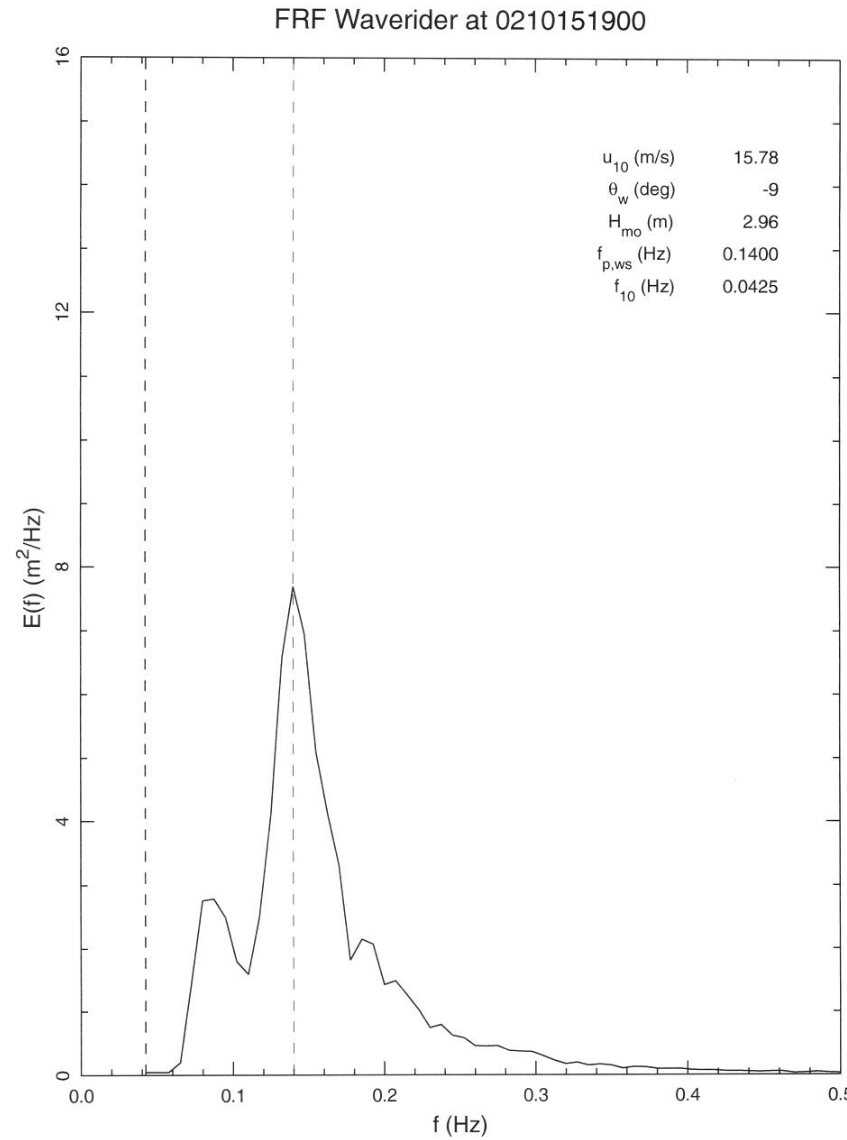

FRF Waverider at 0210140700
u_{10} (m/s) 14.11
θ_w (deg) 64
H_{mo} (m) 2.56
$f_{p,ws}$ (Hz) 0.1475
f_{10} (Hz) 0.0425
E(f) (m^2/Hz)
f (Hz)


FRF Waverider at 0210141900
E(f) (m²/Hz)
f (Hz)
u10 (m/s) 9.87
θw (deg) 31
Hmo (m) 2.14
fp,ws (Hz) 0.1250
f10 (Hz) 0.1475

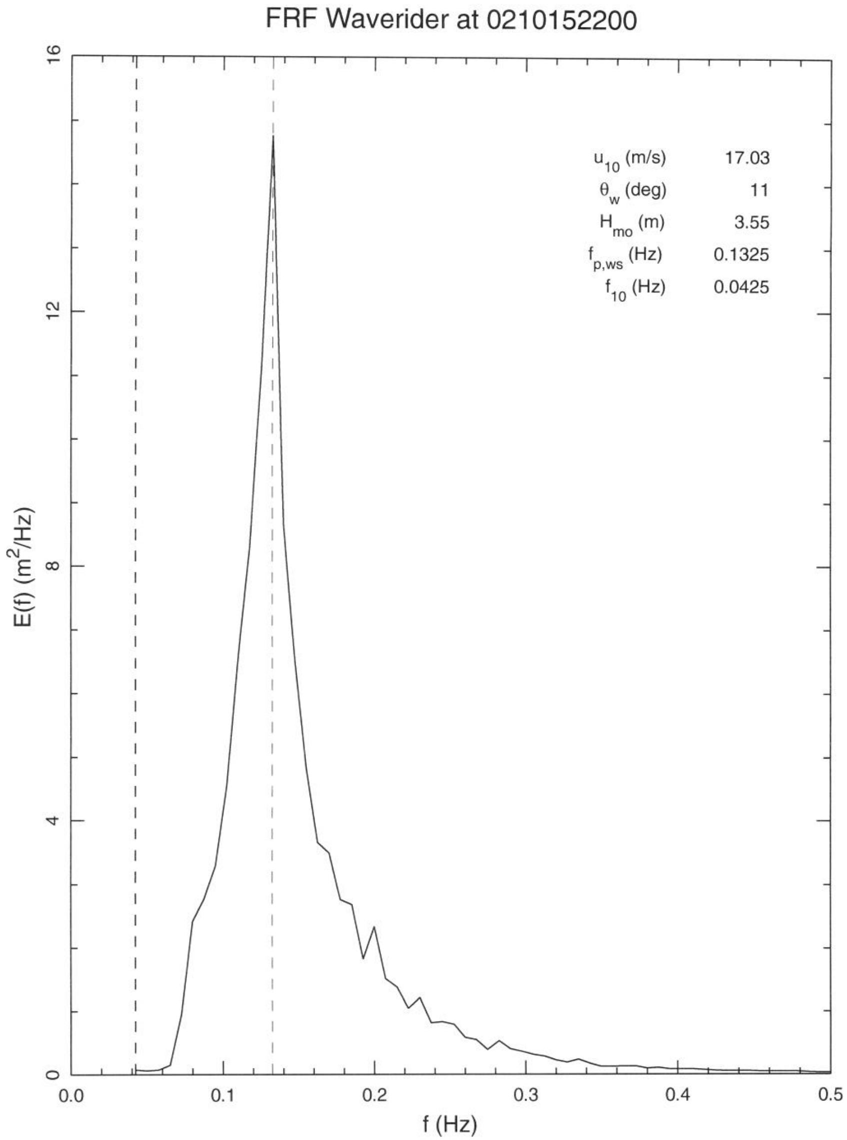

FRF Waverider at 0210151000
E(f) (m²/Hz)
f (Hz)
u10 (m/s) 11.05
θw (deg) 5
Hmo (m) 2.21
fp,ws (Hz) 0.1775
f10 (Hz) 0.1175



FRF Waverider at 0210151300
u_{10} (m/s) 12.96
θ_w (deg) -3
H_{mo} (m) 2.41
$f_{p,ws}$ (Hz) 0.1625
f_{10} (Hz) 0.0425
E(f) (m^2/Hz)
f (Hz)



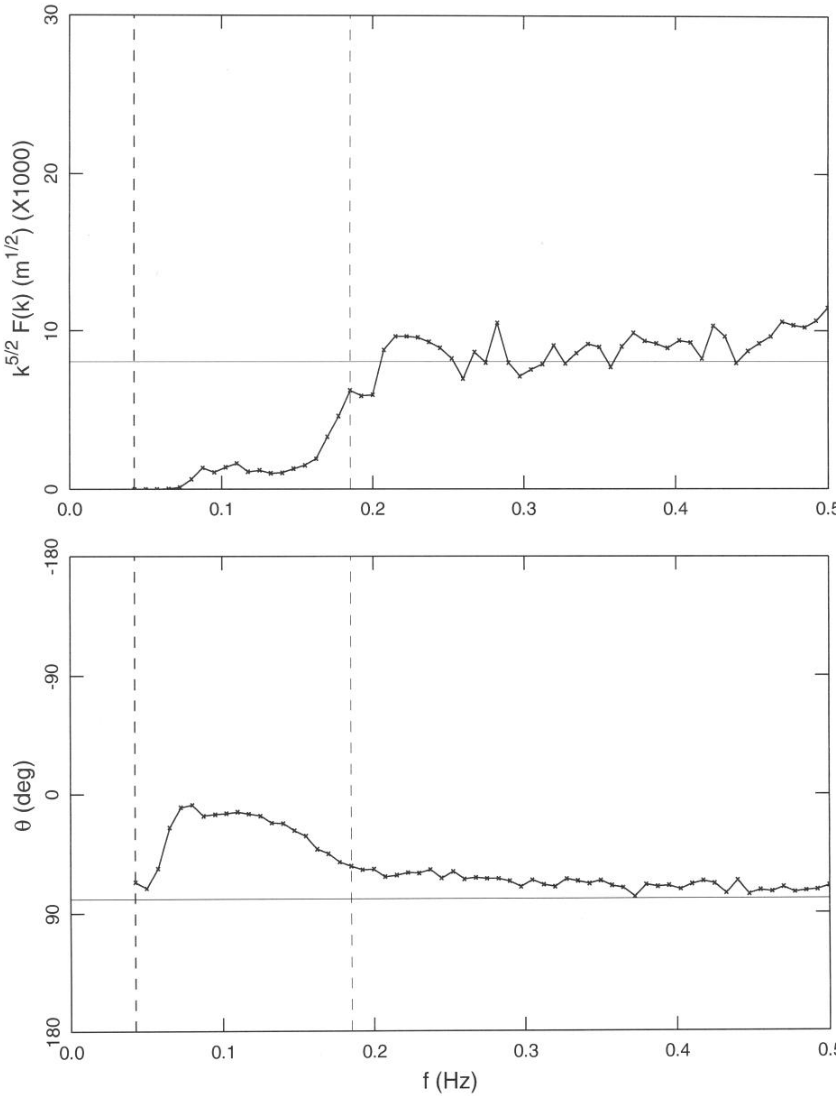
FRF Waverider at 0210151900
u_{10} (m/s) 15.78
θ_w (deg) -9
H_{mo} (m) 2.96
$f_{p,ws}$ (Hz) 0.1400
f_{10} (Hz) 0.0425
E(f) (m^2/Hz)
f (Hz)



FRF Waverider at 0210152200
u_{10} (m/s) 17.03
θ_w (deg) 11
H_{mo} (m) 3.55
$f_{p,ws}$ (Hz) 0.1325
f_{10} (Hz) 0.0425
E(f) (m^2/Hz)
f (Hz)
0
4
8
12
16
0.0
0.1
0.2
0.3
0.4
0.5

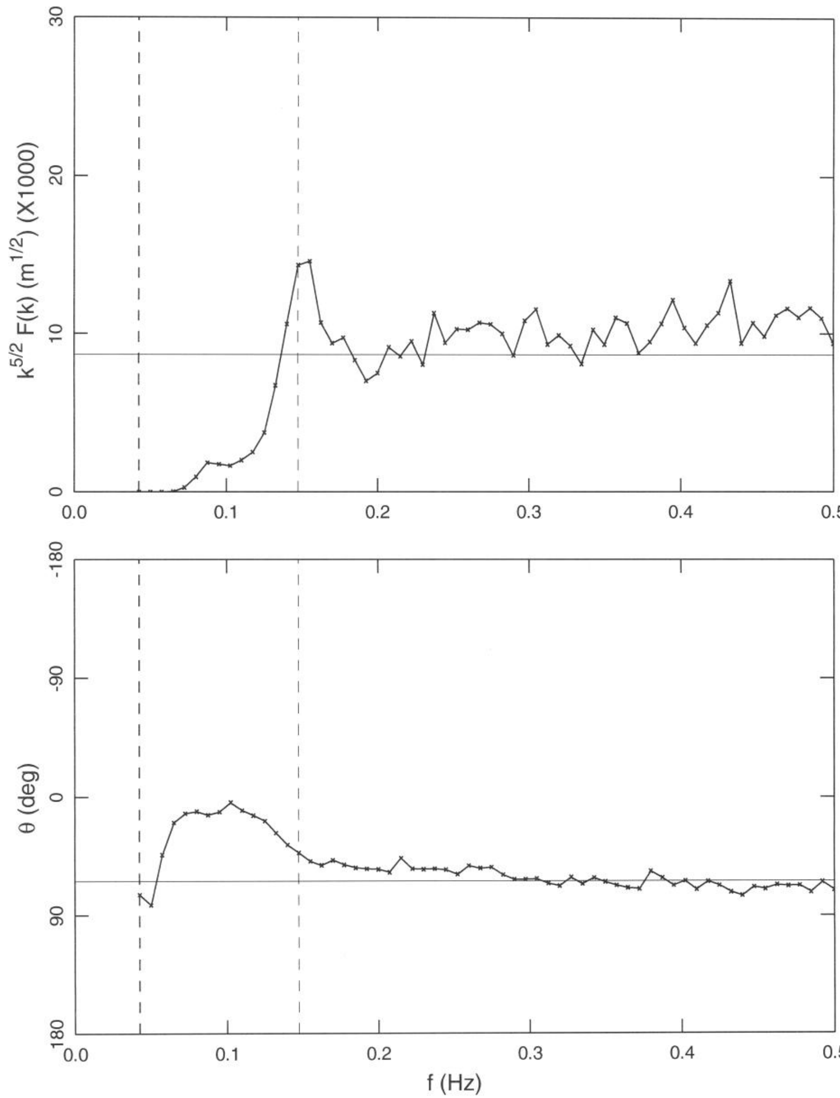
FRF Waverider at 0210140100

	H_{mo} (m)	$f_{p,ws}$ (Hz)	θ_p (deg)	u_{10} (m/s)	θ_w (deg)	h (m)
WR	1.79	0.1850	17	14.60	80	17.0



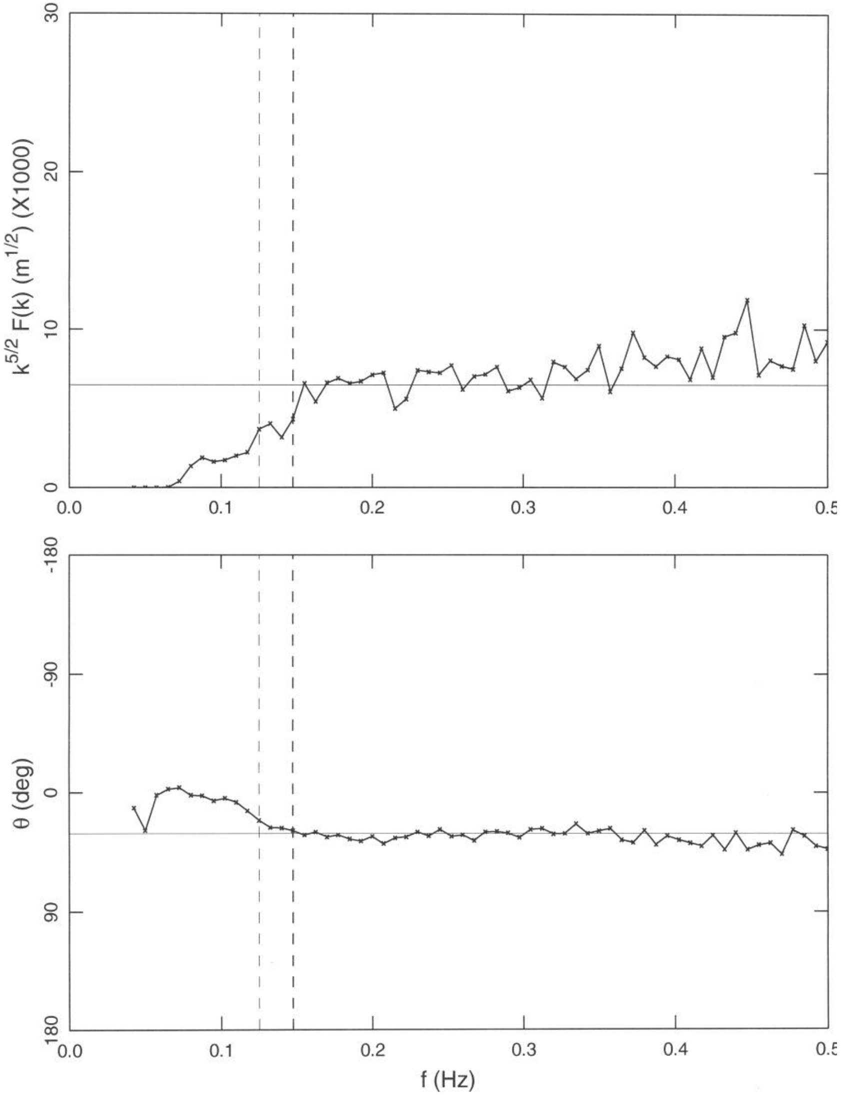
FRF Waverider at 0210140700

	H_{mo} (m)	$f_{p,ws}$ (Hz)	θ_p (deg)	u_{10} (m/s)	θ_w (deg)	h (m)
WR	2.56	0.1475	43	14.11	64	17.0



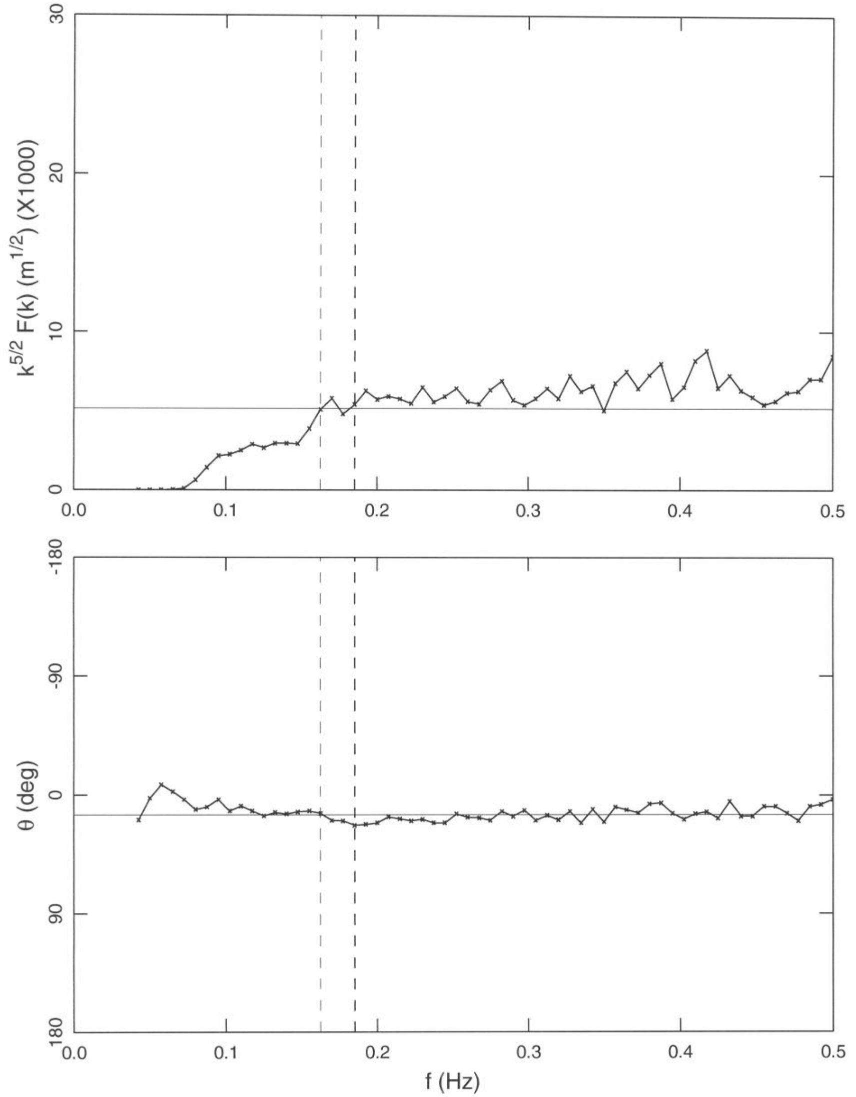
FRF Waverider at 0210141900

	H_{mo} (m)	$f_{p,ws}$ (Hz)	θ_p (deg)	u_{10} (m/s)	θ_w (deg)	h (m)
WR	2.14	0.1250	2	9.87	31	17.0



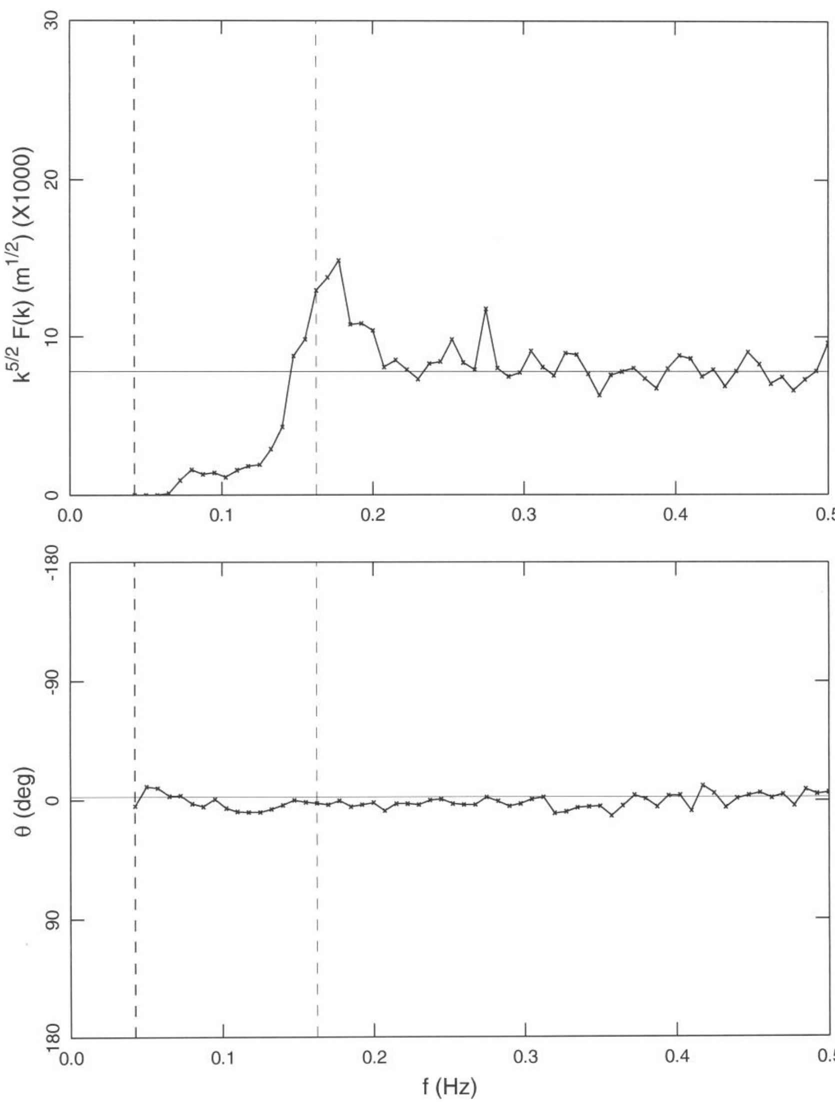
FRF Waverider at 0210150100

	H_{mo} (m)	$f_{p,ws}$ (Hz)	θ_p (deg)	u_{10} (m/s)	θ_w (deg)	h (m)
WR	1.97	0.1625	4	8.43	16	17.0



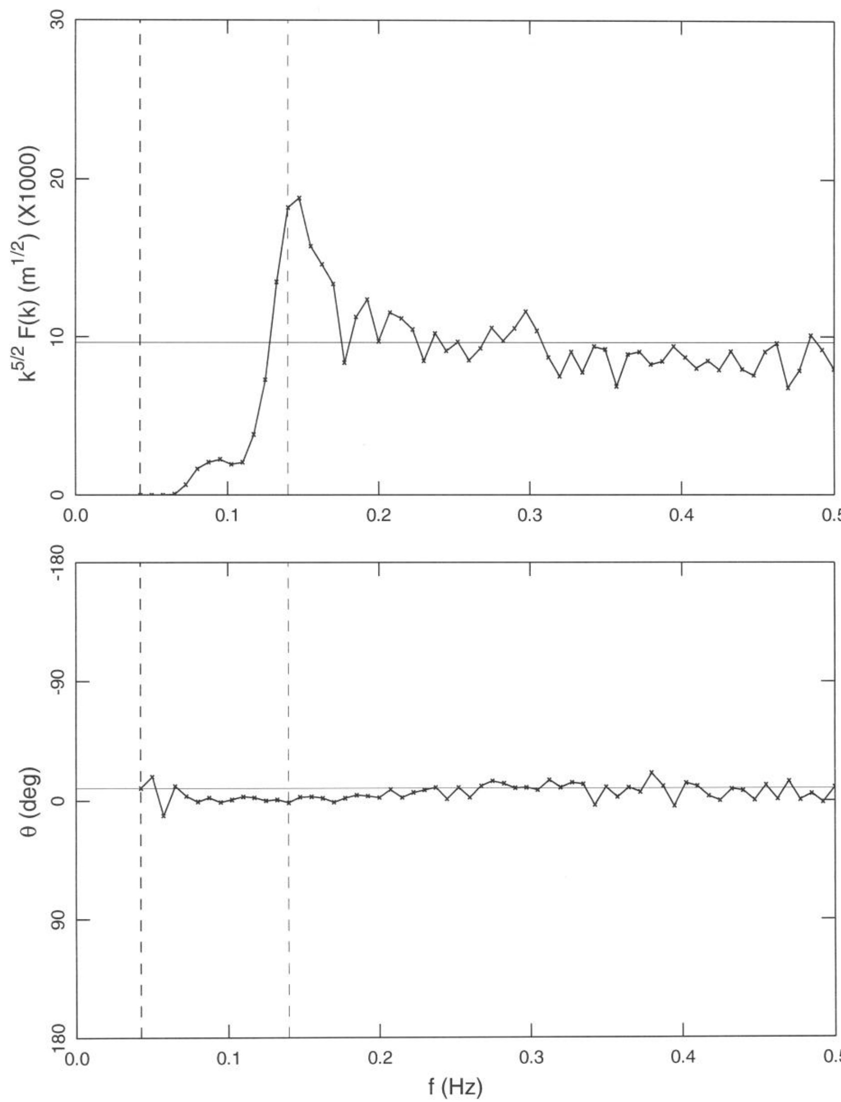
FRF Waverider at 0210151300

	H_{mo} (m)	$f_{p,ws}$ (Hz)	θ_p (deg)	u_{10} (m/s)	θ_w (deg)	h (m)
WR	2.41	0.1625	2	12.96	-3	17.0



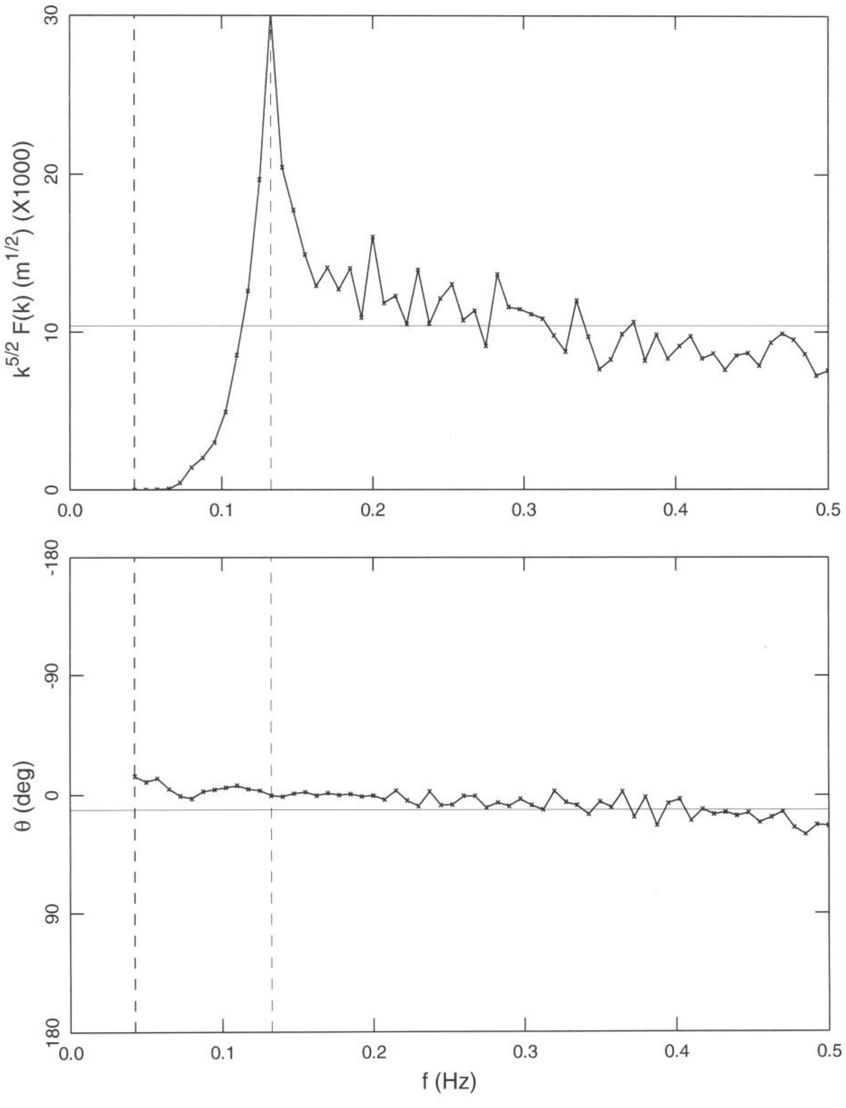
FRF Waverider at 0210151900

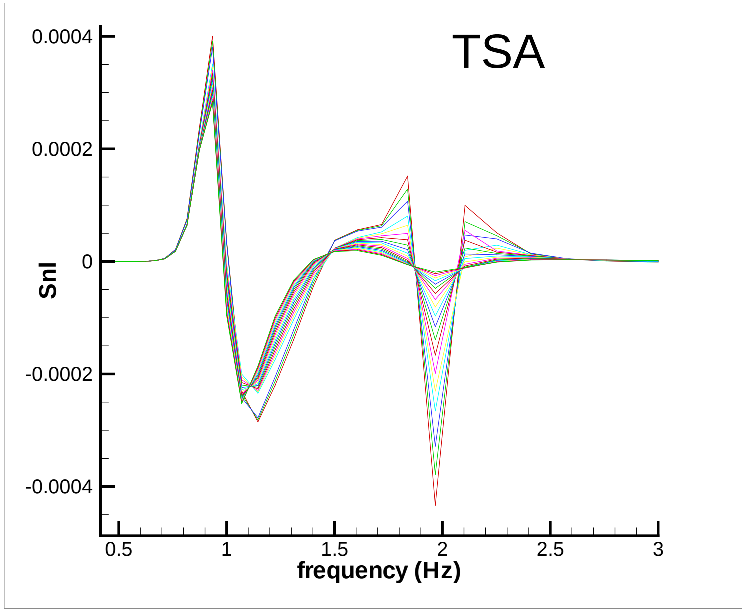
	H_{mo} (m)	$f_{p,ws}$ (Hz)	θ_p (deg)	u_{10} (m/s)	θ_w (deg)	h (m)
WR	2.96	0.1400	2	15.78	-9	17.0

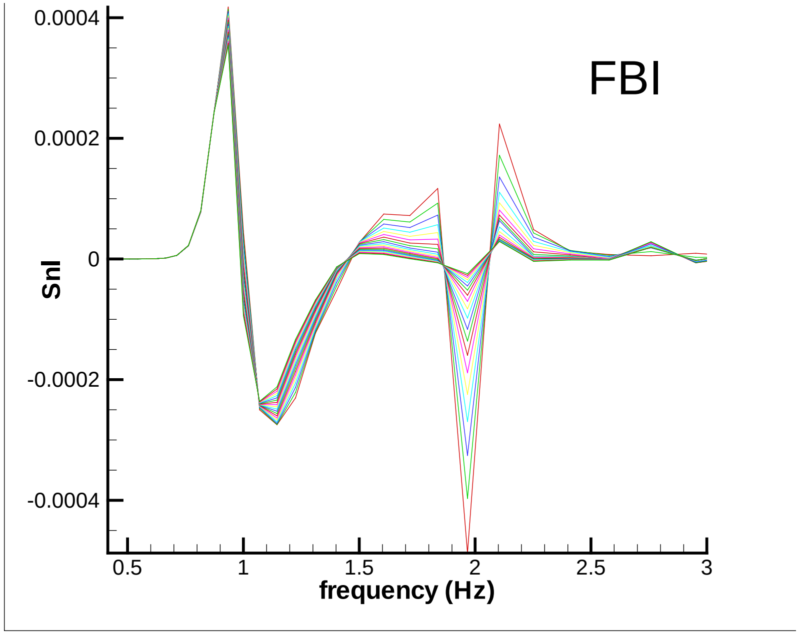


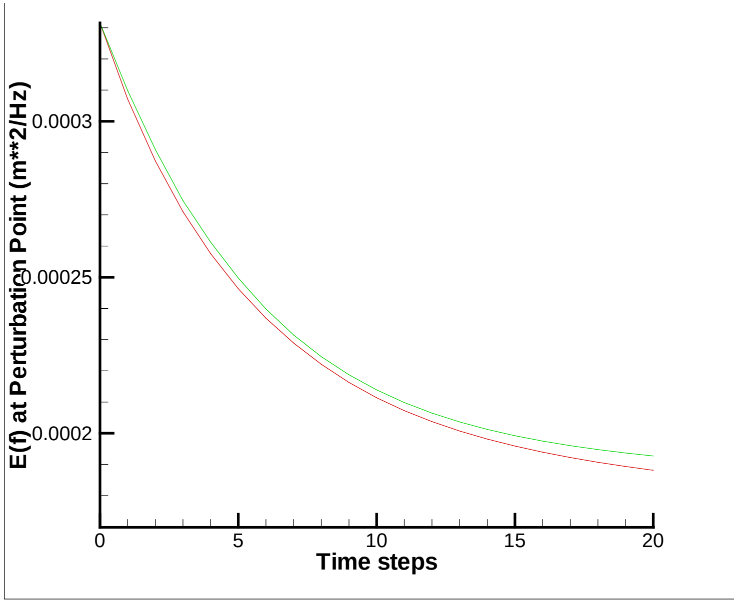
FRF Waverider at 0210152200

	H_{mo} (m)	$f_{p,ws}$ (Hz)	θ_p (deg)	u_{10} (m/s)	θ_w (deg)	h (m)
WR	3.55	0.1325	1	17.03	11	17.0









TSA and FBI have essentially the same relaxation rate to a perturbation

Fetch-growth and associated spectra have long been recognized to follow a self-similar pattern.

So, the first scale of the TSA should be able to capture most of the details of SNL during this growth.

2nd term in TSA can focus on more complex situations – which is more along the lines of Hasselmann's concept for wave growth in 1976.

QUESTIONS???