A Comparison of the Discrete Interaction Approximation to the Full Boltzmann Integral

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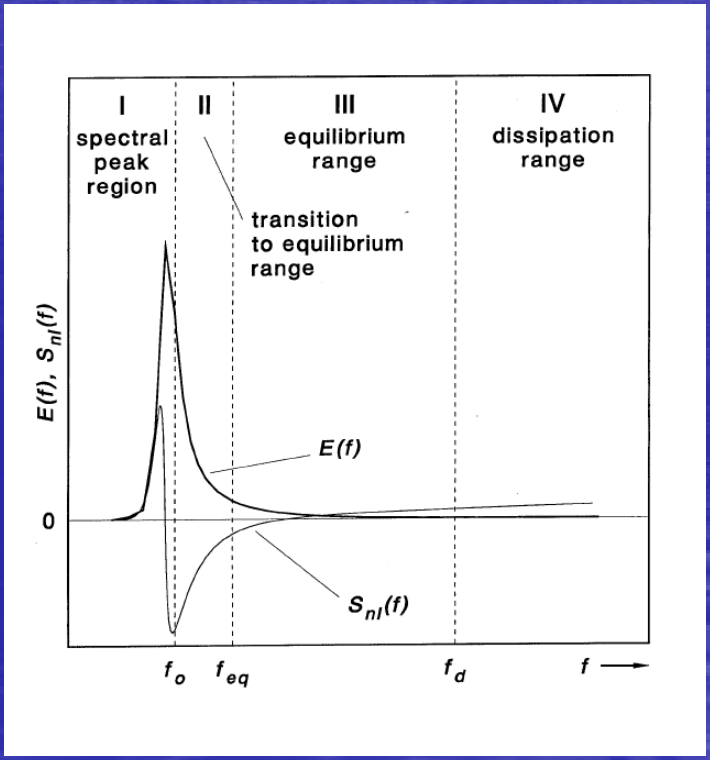
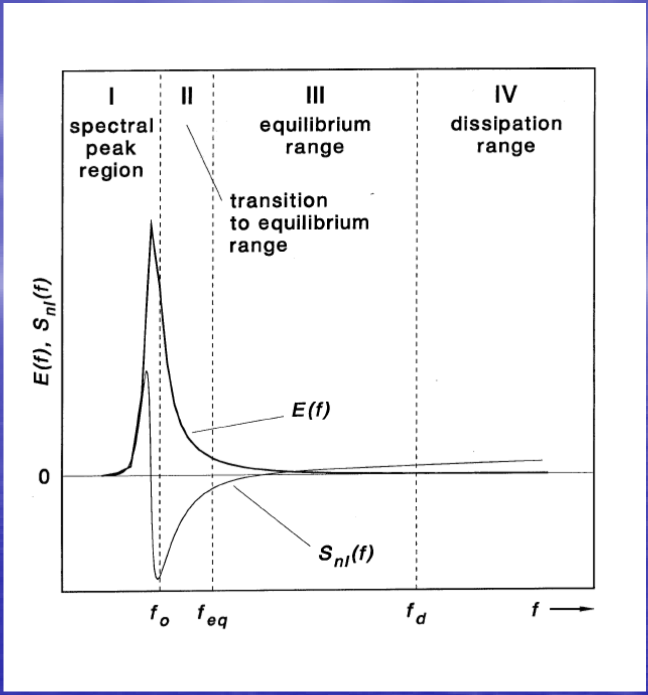


MOTIVATION



- Recent studies have shown that nonlinear interactions can be numerically derived directly from Euler equations.
- Many characteristics of the spectra are consistent with a strong role played by Snl.
- The DIA and many variations of this method for estimating Snl seem to have been accepted with little or no effective validation.
- This paper will examine its suitability for wave models and postulate some guides for what should be required in a properly posed Snl term.

APPROACH:

- Review the role of Snl in the source term balance in 3G and 2G models
- Show theoretical and geometric interpretation of interaction space for Snl
- Show theoretical and geometric relationship and lack of equivalence between DIA and FBI
- Show some results of comparisons of DIA to FBI for hypothetical and observed spectra
- List some theoretical and practical demands that should be placed on Snl approximations

CONCLUSIONS

- General integration space for SnI is in 3 dimensions
- DIA's 4 points from a 3-D volume is extremely deficient and leads to mis-estimations and instabilities
- Other samples using a small # of points relative to the number of points in the FBI SnI estimate are likely to suffer from the same problems
- Recommended guidelines for an operational SnI representation are given

The governing equation for wave generation, dissipation and propagation has remained the same for many decades

$$\frac{\partial E(f,\theta)}{\partial t} = \sum_{k=1}^{n} S_k(f,\theta) - \vec{c}_g \cdot \nabla E(f,\theta)$$

f	is the frequency of a spectral component
θ	is the propagation angle of a spectral component
$E(f,\theta)$	is the spectral energy density at f,θ
t	is time
S_k	is the kth source term; and
c_g	is the group velocity of waves with frequency, f.

For "simple" wind fields (big and broad) a parametric (SMB) model would probably suffice to give "good" results and errors could be seen in terms of wind errors.

For complex situations parametric and empirically tuned codes can deviate strongly. Hurricanes and coastal areas!

- **In 1G models, the number of source terms "n" was assumed to be 2, wind input and dissipation**
- **In 2G models, Snl was added and "n" was assumed to be 3**
- **In 3G models, "n" remained 3, but the method of solution changed to a detailed balance form and Snl $\longrightarrow$ DIA**

Webb (1978) form for nonlinear interactions: basis for WRT method. Masuda (1980) derived variation of this same integral.

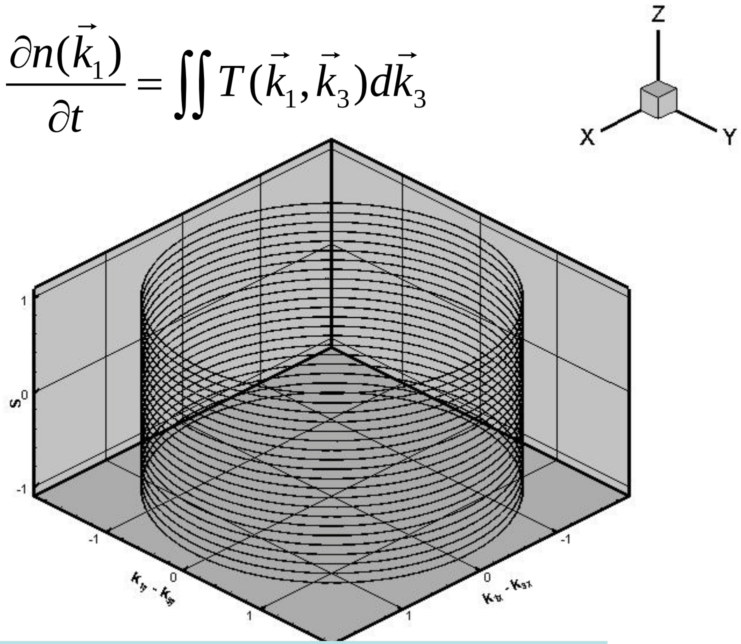
$$I = \frac{\partial n(\vec{k}_1)}{\partial t}$$

$$= \iint\iint\iint T(\vec{k}_1,\vec{k}_3)\delta(\sum s_i k_i)\delta(\sum s_i \omega)dk_2 dk_3 dk_4$$

Initial 6-dimension interaction volume

which can be reduced (Webb, 1978)

$$\frac{\partial n(\vec{k}_1)}{\partial t} = \iint T(\vec{k}_1,\vec{k}_3)d\vec{k}_3$$



Geometric depiction of 3-D interaction space (volume)

$$T(\vec{k}_1,\vec{k}_3) = 2\oint D^3 G ds$$

with n_i implying $n(\vec{k}_i)$

$$D^3 = n_1 n_3(n_4 - n_2) + n_2 n_4(n_3 - n_1)$$

where and

$$G = C(k_1,k_2,k_3,k_4)\left|\frac{\partial W}{\partial k_n}\right|^{-1}$$

$$\times H(|\vec{k}_1 - \vec{k}_4| - |\vec{k}_1 - \vec{k}_3|)d$$

Ignoring the details of this integral we see that the Webb integration is carried out over 3 dimensions

2 dimensions in k3
1 dimension around the locus "s"

Masuda form also contains 3-D

After trying a number of statistical and other approximations Hasselmann et al. (1985) determined that the best method to retain the same number of degrees of freedom in Snl was to simplify the integral –

by adding another delta function into the argument

$$G = C(k_1, k_2, k_3, k_4)\left|\frac{\partial W}{\partial k_n}\right|^{-1} H(|\vec{k}_1 - \vec{k}_4| - |\vec{k}_1 - \vec{k}_3|)\underline{\delta(\vec{k}_1 - \vec{k}_2)}$$

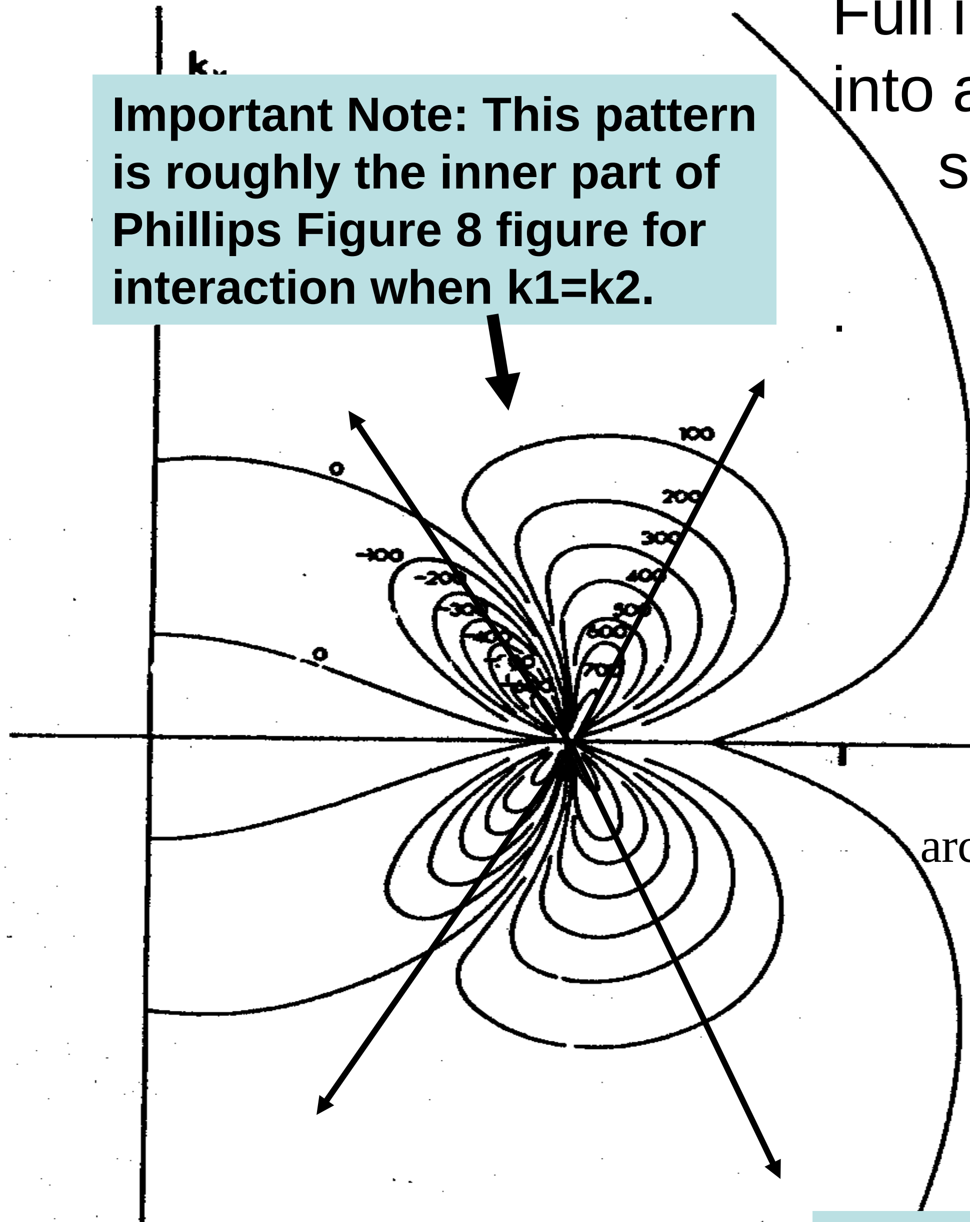
"New" delta function in k1-k2

The argument was essentially based on the behavior of the density function being largest when 2 of the 4 interacting waves were near the spectral peak.

This, of course, is not equal to the full integral and had to have an empirical multiplier.

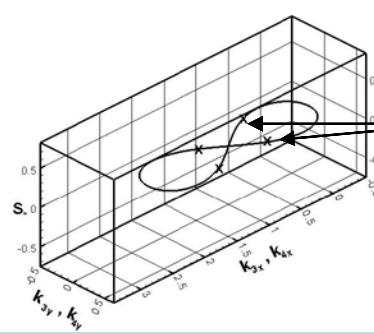
Full integral estimate for transfers into and out of a location near the spectral peak (Webb, 1978)

Important Note: This pattern is roughly the inner part of Phillips Figure 8 figure for interaction when k1=k2.



Transfers are very comparable to the results of Longuet-Higgins and Fox (1975) based on Davey and Stewartson (1974) work on a 3-dimensional wave packet instability. Note asymptotic angles of transfer maxima at

$$\arctan\left(\frac{1}{\sqrt{2}}\right) = \text{asymptote of "Figure 8" pattern}$$



Little x's are the quadruplets

Figure 8 is a line imbedded in 3-D interaction volume. Quadruplets reduce this to only 4 pts

Contours of $T(\underline{k}_1, \underline{k}_3)$

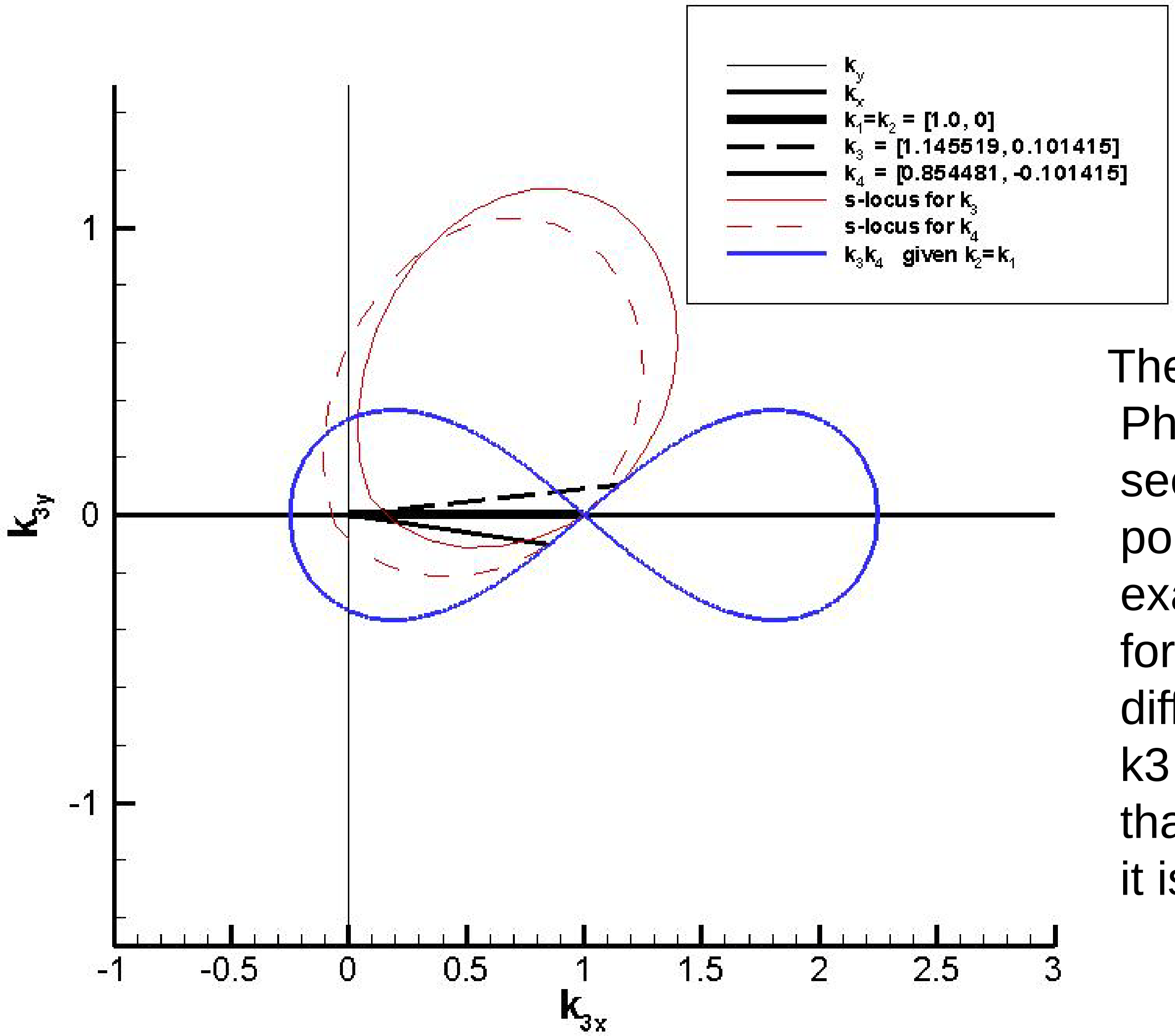
The argument was essentially based on the behavior of the density function being largest when 2 of the 4 interacting waves were near the spectral peak.

This, of course, is not equal to the full integral and had to have an empirical multiplier, formally given by:

$$Z_{DIA}(\vec{k}_1) = \frac{\iint 2\oint D^3 G \delta(\vec{k}_1 - \vec{k}_2) ds d\vec{k}_3}{\iint 2\oint D^3 G ds d\vec{k}_3}$$

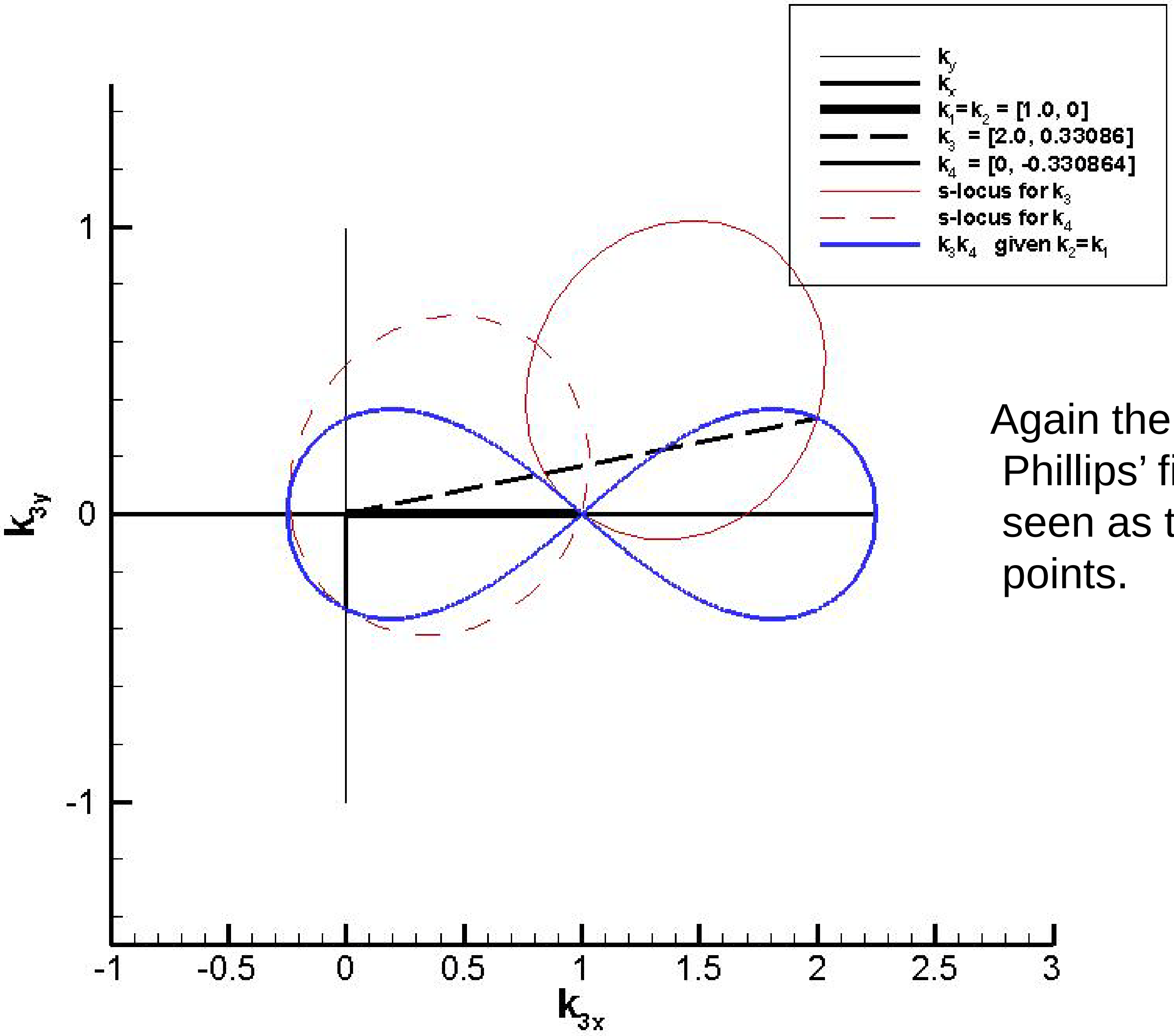
As written this would be formally equal to zero, since the integration volume is equal to zero, but in the discretized form incremental phase volumes are retained.

Note the Z is actually not a single constant over the entire spectrum.

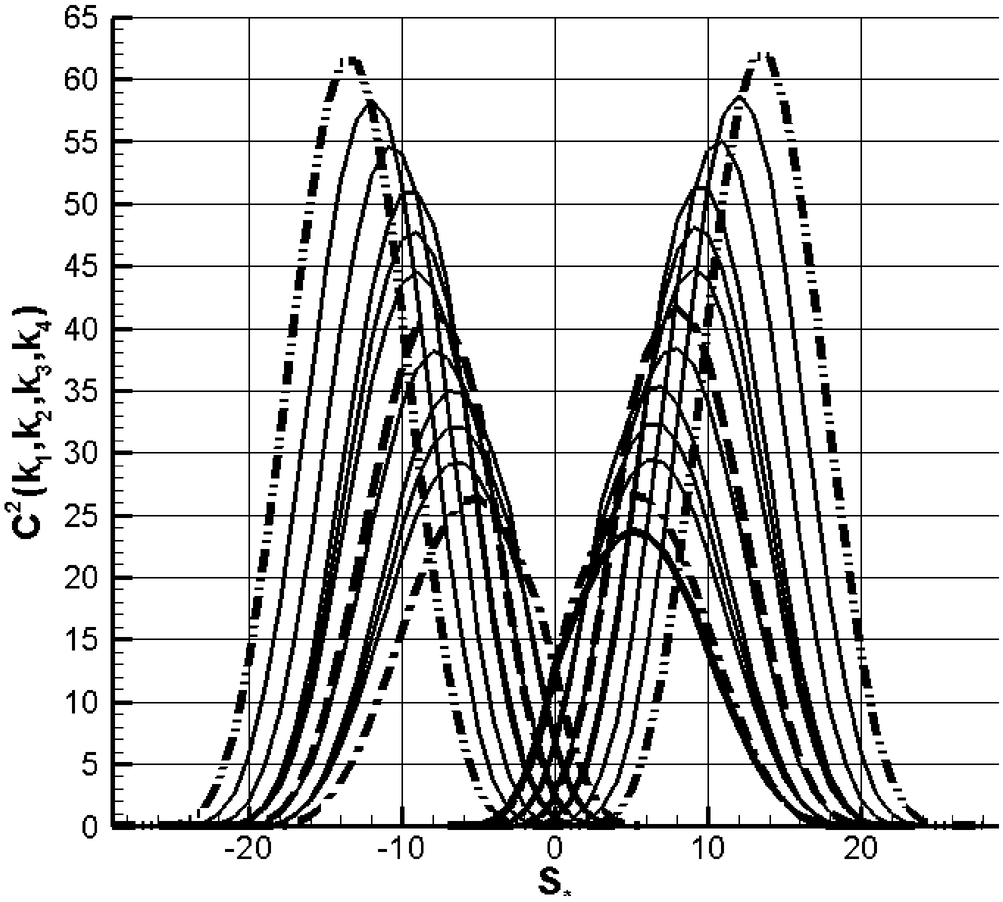


The two points on the Phillips' figure-8 can be seen as two "matched" points such that they exactly fit the WRT formulation for the differences between the k3 and k4 loci – except that typically for the WRT it is the k2 and k4 loci.

As seen here, any figure-8 (and quadruplets) can be viewed in terms of a special case of the WRT integration method, but is certainly not sufficient to approximate the entire WRT (or Masuda) integral.



Again the points on the Phillips' figure-8 can be seen as two "matched" points.



Did the quadruplets in the original DIA represent particularly large and strongly coupled wave interactions?

Here the "0" point is a point along the s-locus.

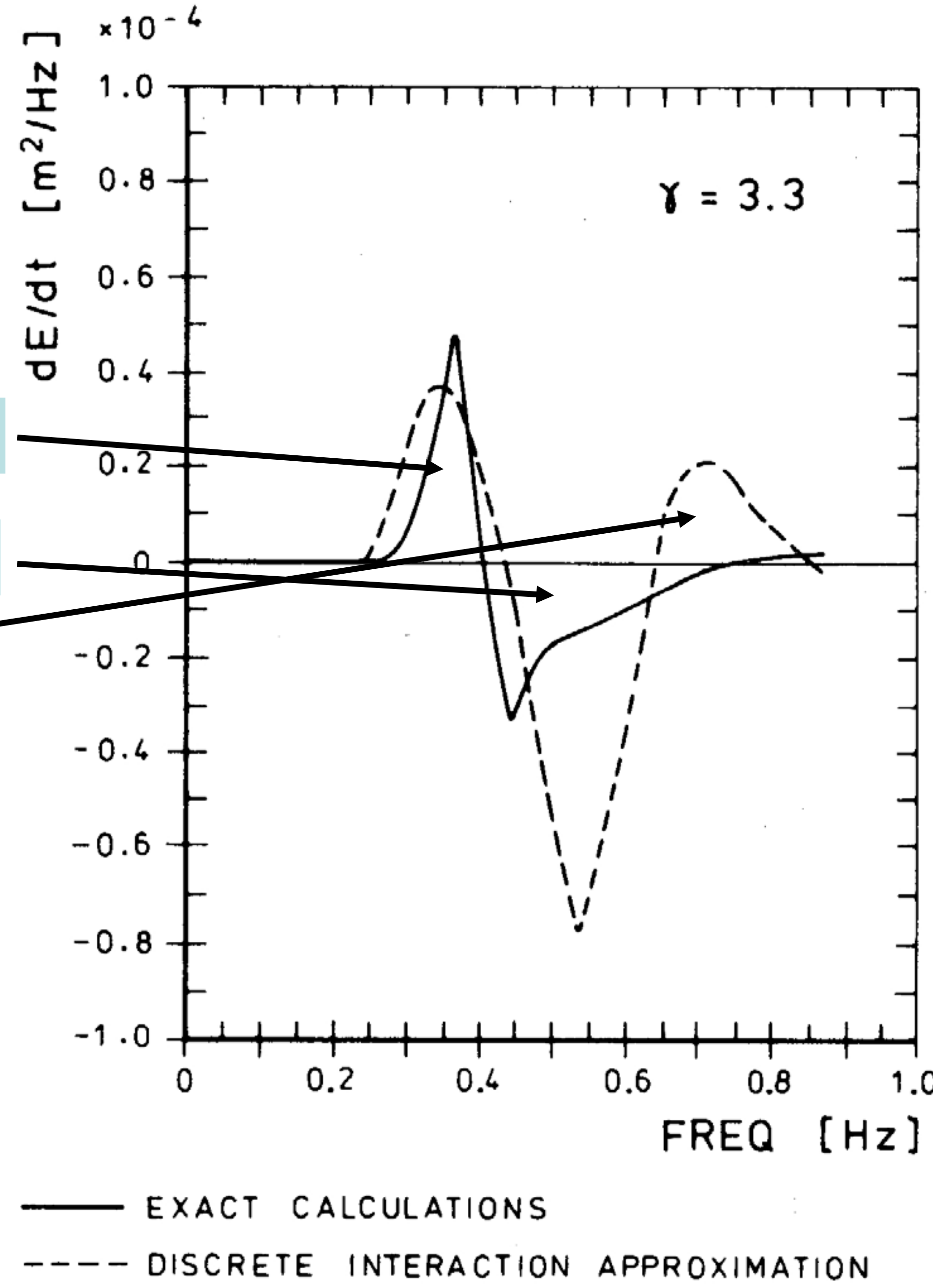
Analogue of using only 4 pts for Snl is very much like estimating amount of gold within a volume of mountain from only 4 point samples

DIA is parametrically calibrated to match only a single quantity – the total energy transferred to the forward face of the spectrum.
but:

Calibrated to “match” here

Creates a mis-match here

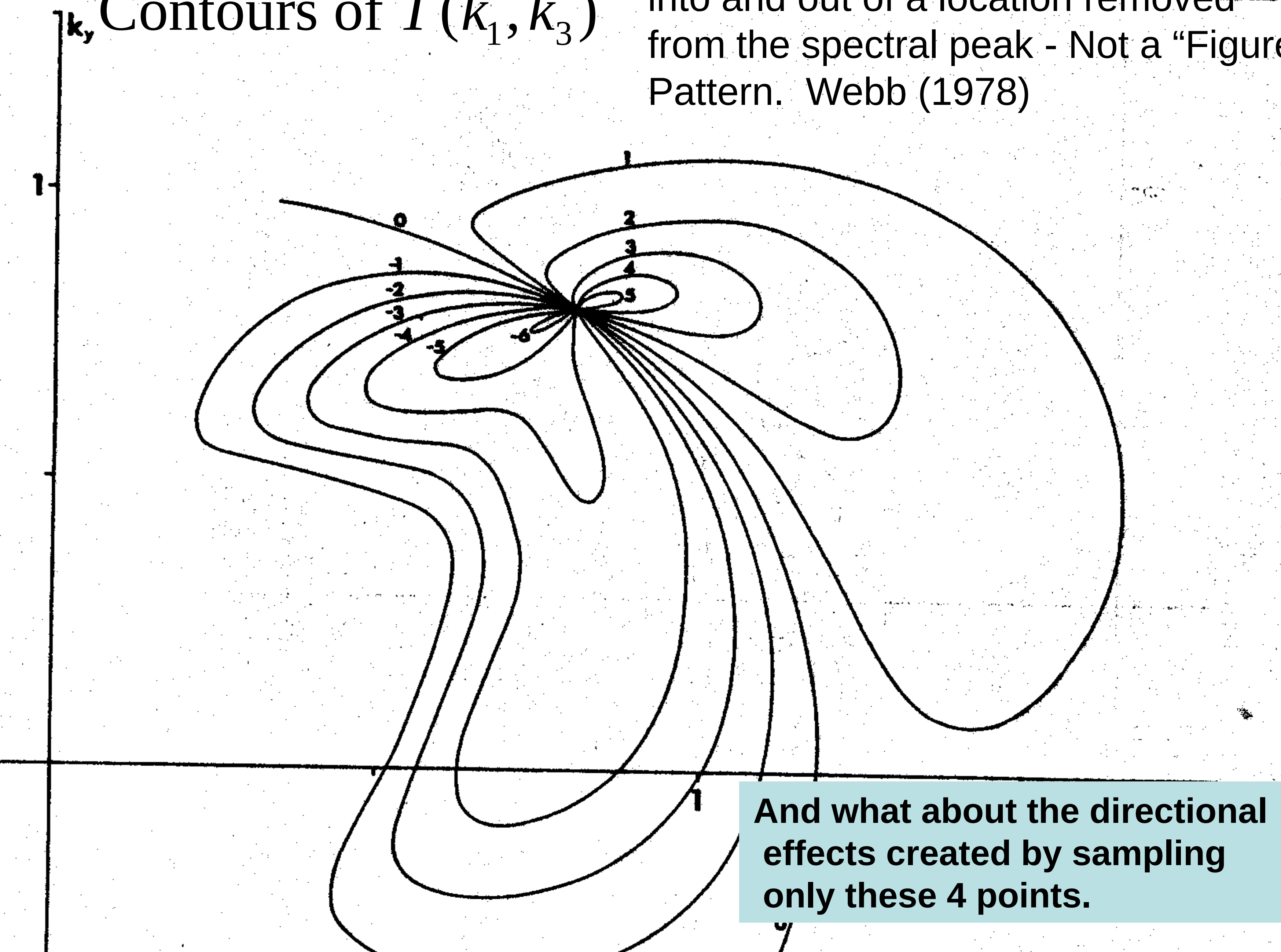
And here



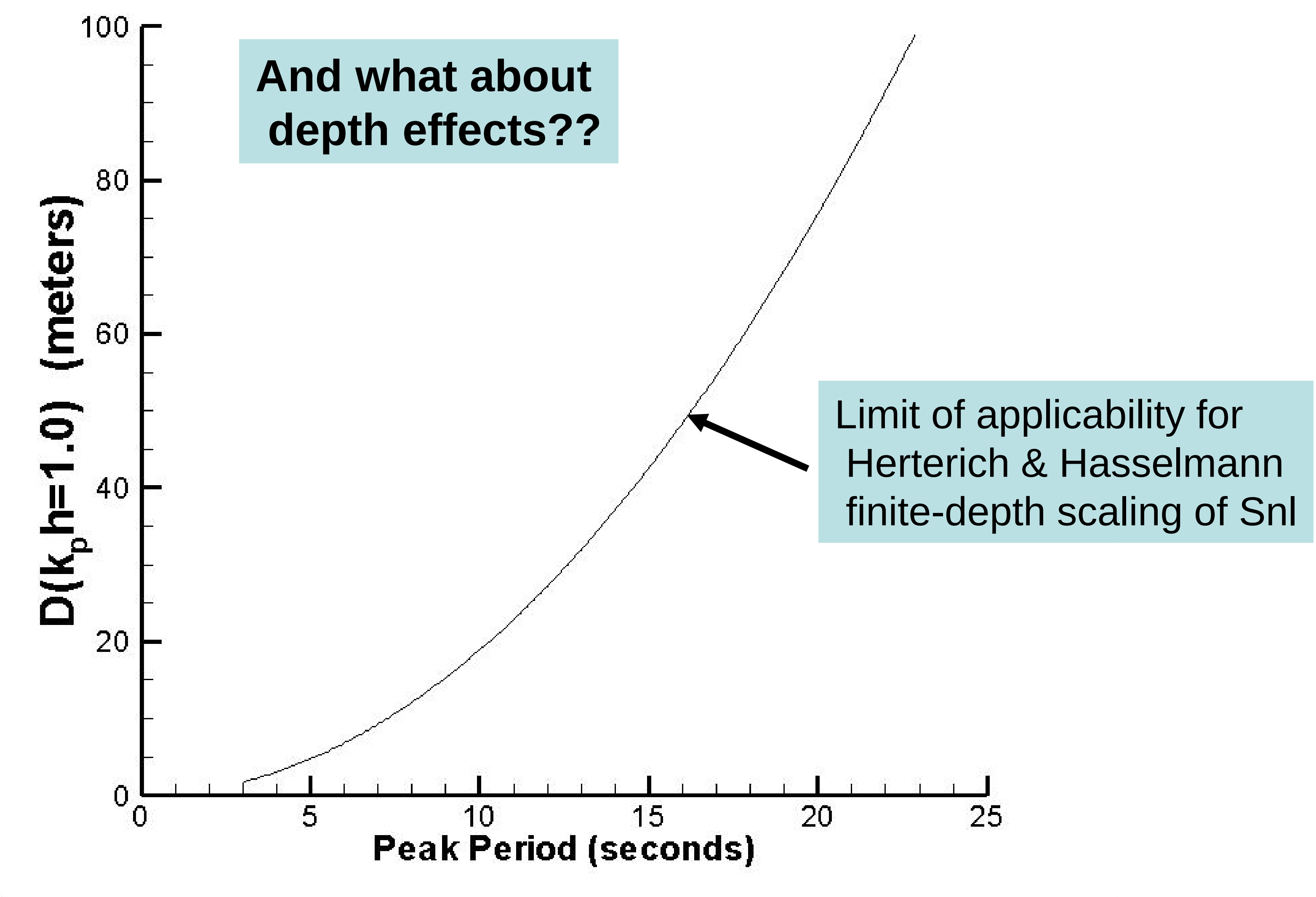
Fluxes from the DIA do not force an f^{-4} equilibrium range

Contours of $T(k_1, k_3)$

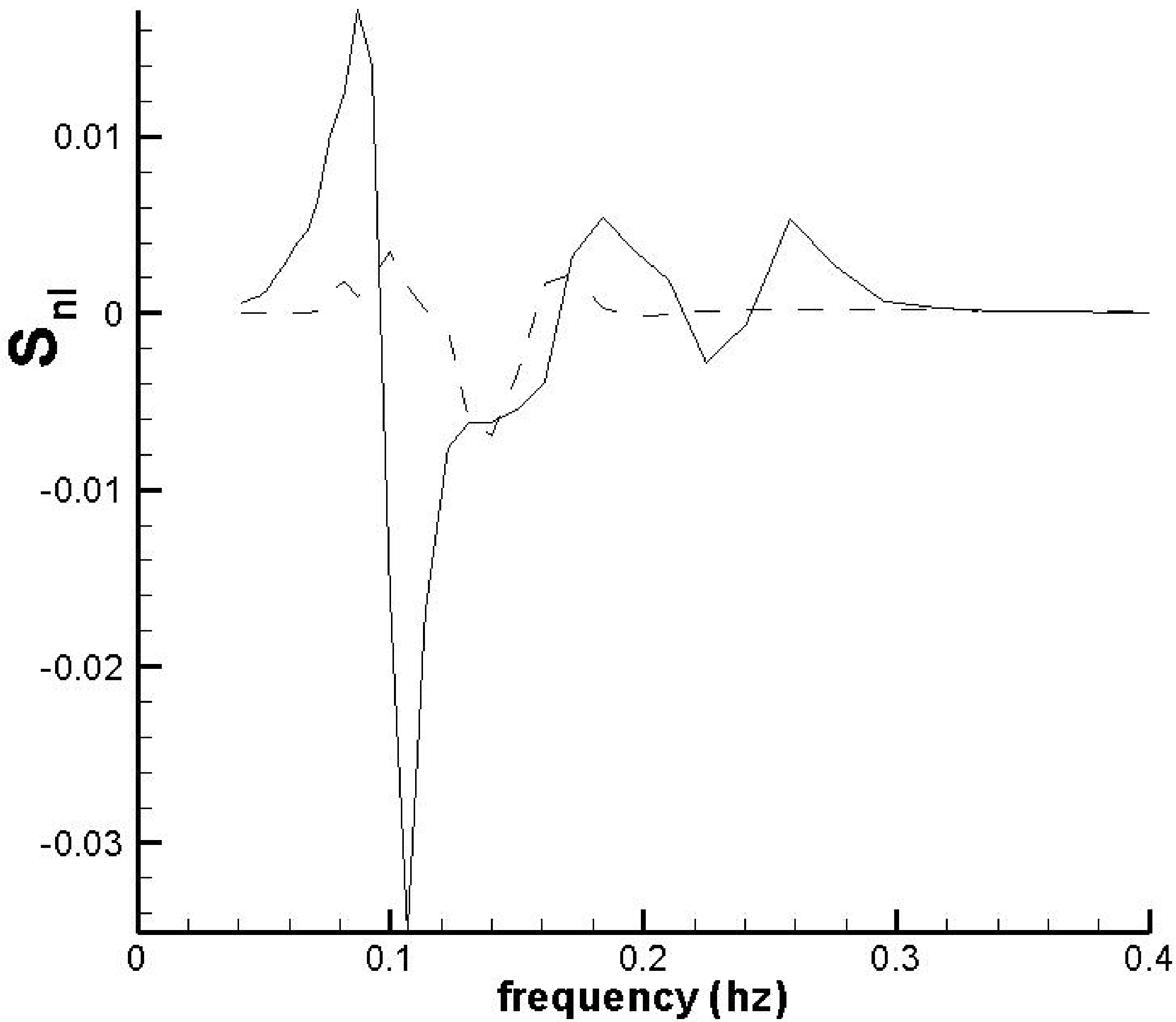
Full integral estimate for transfers into and out of a location removed from the spectral peak - Not a “Figure 8” Pattern. Webb (1978)



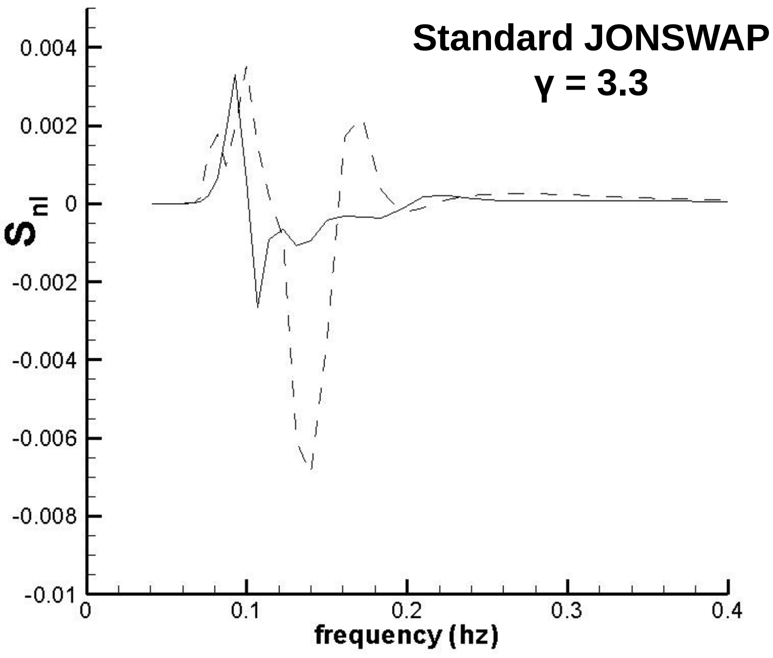
And what about the directional effects created by sampling only these 4 points.

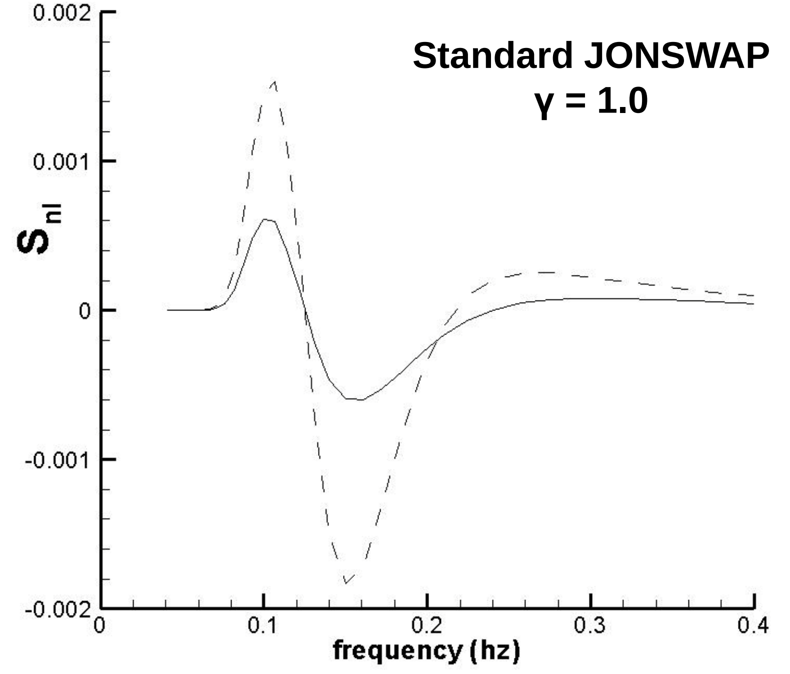


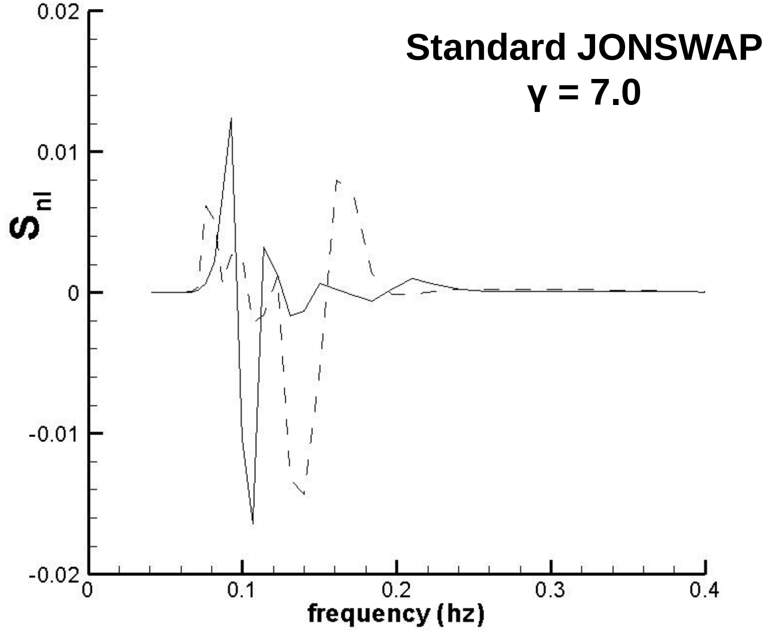
The accuracy of the present DIA approximation for Snl ($k_p h$ =1) is very restrictive for coastal applications! What happens if you violate this limit?



Comparison of a scaled DIA calculation and an actual full-integral, finite-depth calculation for the case of kph=0.7 (JONSWAP spectrum with a peak period of 10 seconds in depth of 10.5 meters). The dashed line is the Herterich and Hasselmann (1980) scaled DIA and the solid line is the finite-depth, full integral.



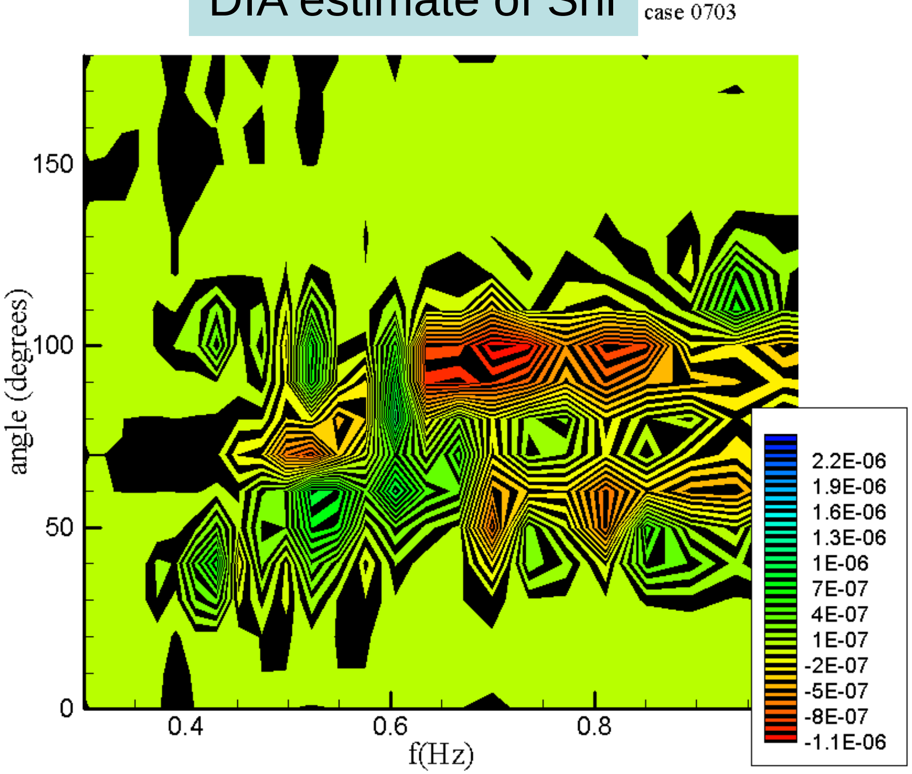




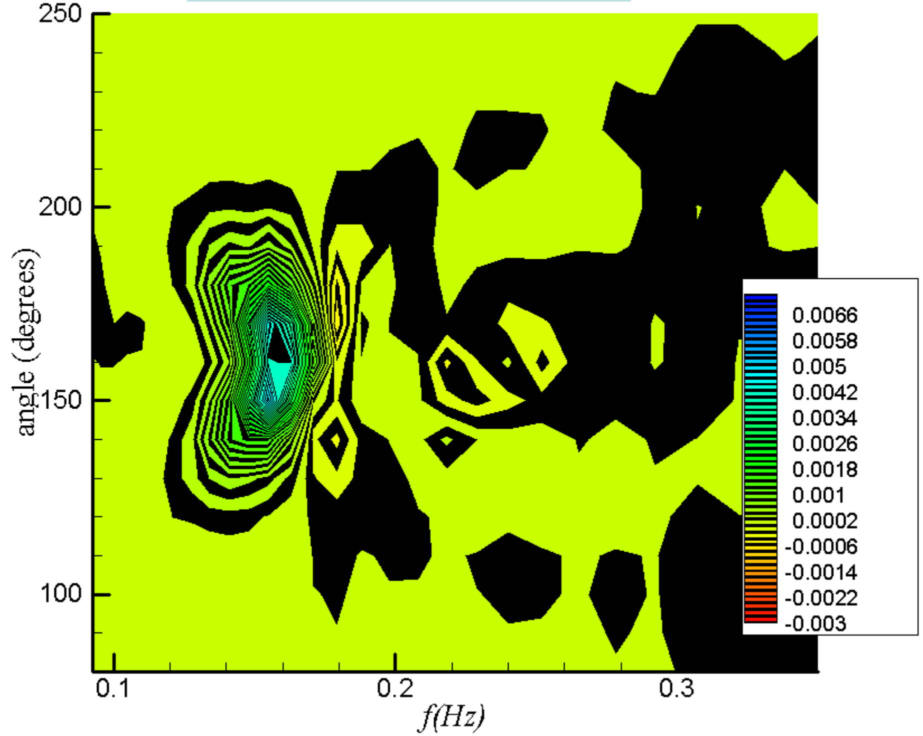
Even for simple parametric spectra in deep water this approximation is not good.

DIA performance for JONSWAP spectra with selected peakedness values. Solid line is full integral. Dashed line is DIA. This does not provide a consistent amount of energy to forward face of spectrum.

DIA estimate of Snl



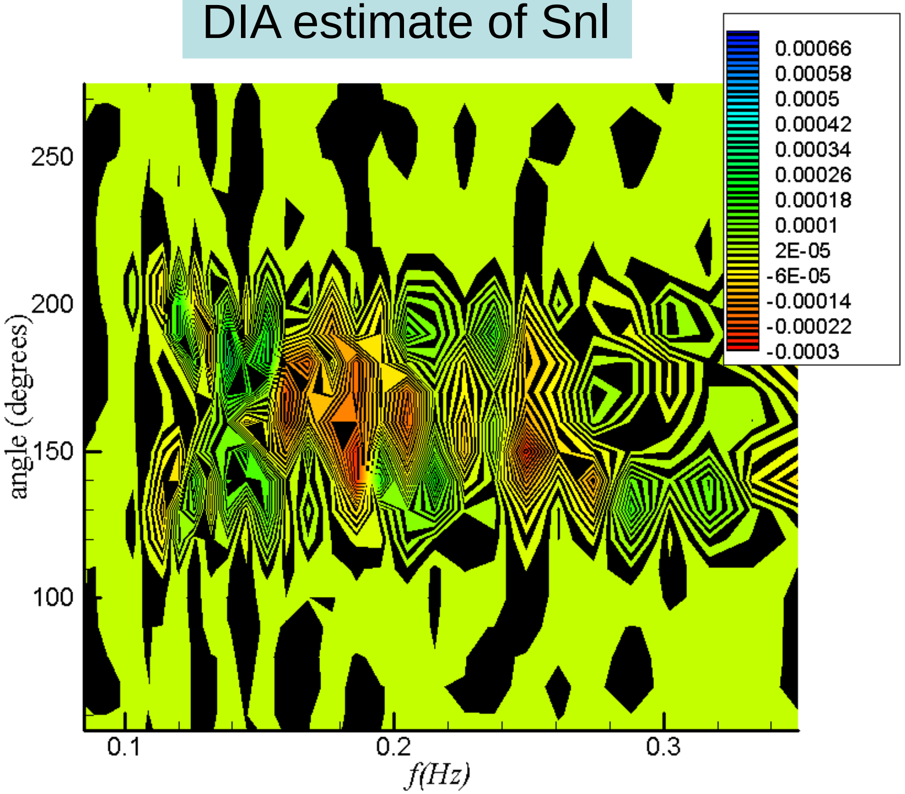
FBI estimate of Snl



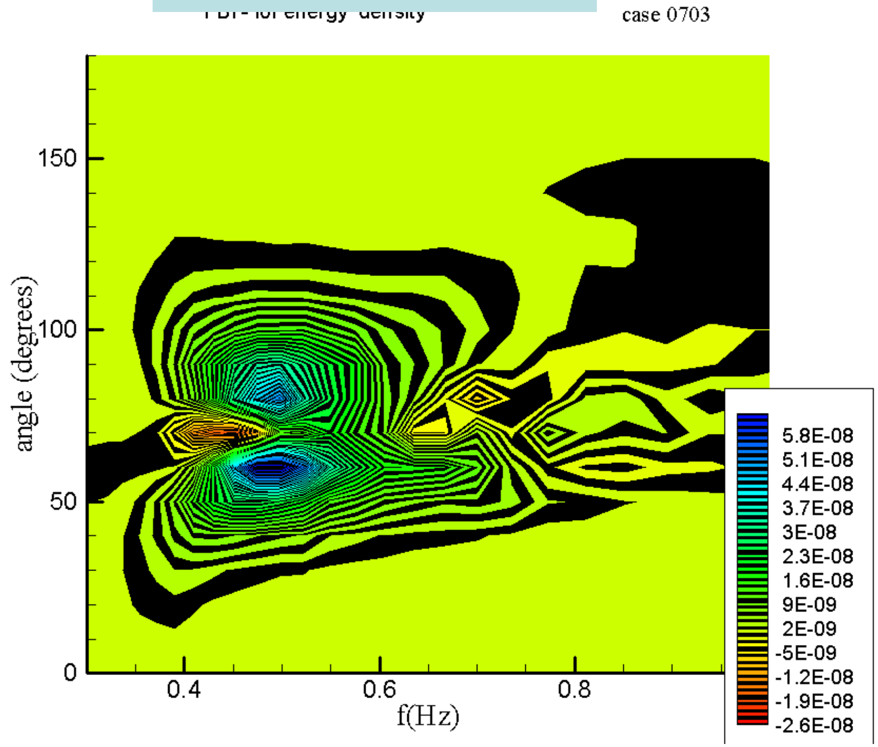
Comparison of DIA and FBI for directionally bimodal Currituck Sound Spectrum

(Including all the spectral warts and blemishes)

DIA estimate of Snl



FBI estimate of Snl



Comparison of DIA and FBI for Waverider spectrum from site in 17 m depth off the coast of Duck, NC

(Including all the spectral warts and blemishes)

A physics-based estimate for Snl should:

1. conserve constants of motion (action, energy, and momentum); otherwise, the need for spurious additional source terms will undoubtedly arise.
2. produce the correct fluxes of action, energy and momentum through the spectrum in order to allow the spectral shapes to evolve in a proper fashion.
3. yield a response to a perturbation in the spectral densities that is quantitatively close to the FBI solution.
4. produce estimates of Snl that deviate from the FBI less than some small percentage, perhaps 15%, or so.
5. be accurately adaptable to coastal water depths.
6. In combination with reasonably posed additional source terms produce dimensionless energy and peak frequency growth rates with fetch up to full development in agreement with field studies.
7. In combination with reasonably posed additional source terms produce directionally integrated spectra and directional distributions of energy consistent with field studies.

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QUESTIONS???????