A Comparison of the Discrete Interaction Approximation to the Full Boltzmann Integral for Wave-Wave Interactions in Wind-Driven Seas

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1. Introduction

Since the 1970's, it has been widely accepted that the effects of wave generation, propagation, and decay can be represented in terms of the so-called radiative transfer equation (RTE),

$$1.\quad \frac{\partial E(f,\theta)}{\partial t} = \sum_{k=1}^{n} S_k(f,\theta) - \vec{c}_g \cdot \nabla E(f,\theta)$$

where

f	is the frequency of a spectral component
θ	is the propagation angle of a spectral component
$E(f,\theta)$	is the spectral energy density at f,θ
t	is time
S_k	is the kth source term; and
c_g	is the group velocity of waves with frequency, f.

Also since that time, it has been widely accepted that three source terms are required to represent the important processes in the first term on the right hand side of equation 1: wind input (S_{in}), nonlinear wave-wave interactions (S_{nl}), and wave breaking/dissipation (S_{ds}). Before the mid-1970's, First-Generation models assumed that only wind input and wave dissipation/breaking were significant to the generation and dissipation of waves. Second-Generation models added the nonlinear source term into equation 1, reflecting a transition in our understanding of the physics of wave generation/decay and spectral evolution in nature.

In spite of general acceptance of S_{nl} as an important element in wave modeling (Young and Van Vleddar, 1993; Komen *et al.*, 1994; Cavaleri *et al*, 2007), the nonlinear wave-wave interaction source term has remained a somewhat mystical factor within wave models for many researchers. Probably because of some remaining skepticism concerning this term, different modelers even today continue to use representations that vary significantly from each other in operational models, which is truly quite surprising

since this is the only source term of the three that can actually be derived from first principles with no recourse to empirical coefficients. However, within the last decade or so, several excellent studies using theoretical methods and direct numerical simulations have clearly demonstrated that 1) wave-wave interactions are critical to the evolution of wave spectra and 2) these interactions are well approximated by the integral developed by Hasselmann in 1962 (Tanaka, 2001a, 2001b, 2007; Korotkevich *et al.*, 2007) Additional studies have demonstrated that the development of many spectral characteristics predicted by these nonlinear interactions (such as the f^{-4} equilibrium range and bimodal directional distribution of energy) are very dominant features of observed wave spectra (Young *et* al., 1995; Ewans, 1998; Wang and Hwang, 2001; Resio *et al*., 2004; Long and Resio, 2007). The recent independent verification of the nonlinear source term and the prevalence of features within observed wave spectra consistent with predictions based on nonlinear interaction theories both suggest that it S_{nl} must be accurately quantified within wave models before the other two terms can be properly formulated within these models. In this light, it seems appropriate at this time to take a critical look at the approximations currently used in existing operational models.

In the mid-1980's, Hasselmann *et al.* (1985) argued that it was important to solve equation 1 in terms of a detailed balance formulation in order to capture critical wave generation-dissipation processes in complex situations. This established the foundation for third-generation models. However, it should be noted that third-generation models retained essentially the same formulation for source terms as embodied within second-generation models. Second-Generation models used detailed balance formulations for their propagation and for their wind input and wave breaking source terms; however, these models used parametric approximations for the nonlinear wave-wave-interaction source term (Barnett, 1968; Ewing, 1971; Resio, 1981). Hasselmann *et al.* (1985) postulated that the number of degrees of freedom in all of the source terms must be as great as the number of degree of freedom used to represent the spectrum within a model; otherwise, artificial constraints on the spectral evolution would be introduced which could significantly affect modeled wave generation and decay rates.

After investigating a number of approaches for approximating S_{nl}, the approximation adopted by Hasselmann *et al.* (1985) was based on a discretized form for a special case of nonlinear interactions due to Phillips' (1958). Resio and Perrie (2008) note that at least five different approaches have been used in attempts to formulation an efficient and accurate approximation to S_{nl}: a) parametric representations; b) local interaction/diffusion operators (LIO's); c) linear combinations of orthogonal functions; d) reduced integral domains; and e) the Discrete Interaction Approximation (DIA). The last of these methods noted here is the approach developed by Hasselmann *et al.* (1985). In addition to these methods, additional methods using statistical methods such as Artificial Neural Networks (ANN's) have been examined (Tolman *et al.*, 2004). Additionally, various other methods which are essentially extensions of the DIA approach have now been developed and are under consideration for implementation within models, such as the MDIA (ref), the EDIA (ref)the SRIAM (ref), and the TSA (Resio and Perrie, 2008; Perrie and Resio, 2009).

The methods mentioned in the previous paragraph can be subdivided into two broad sub-categories; 1) methods which are constrained to conserve energy, action, and momentum and 2) methods which are not constrained to conserve energy, action, and momentum. Given the importance of conservation to the physics of the processes being modeling (which is likely to be exacerbated in shallow water), we decided to omit non-constrained methods and methods inappropriate for detailed-balance calculations (methods which do not retain at least as many degrees of freedom in the approximation to S_{nl} as in the directional spectrum being modeled) from further analysis here. This eliminates parametric methods and statistical methods (both orthogonal function approximations and ANN approximations). Also, since LIO's have not been shown to be effective in operational models these will not be addressed in this paper. This leaves only discretized approximations of the type represented by the DIA, MDIA, EDIA, SRIAM, and TSA.

2. Theoretical Considerations

All of the approximations considered here are based on the Full Boltzmann Integral (Hasselmann, 1962), an integro-differential equation which relates the rate of change of action density in vector wavenumber space [$n(\vec{k})$] to an integral over the entire spectrum

$$2.\quad \frac{\partial n(\vec{k}_1)}{\partial t} = \iint T(\vec{k}_1, \vec{k}_3) d\vec{k}_3$$

where

$$T(\vec{k}_1, \vec{k}_3) = 2\iint D^3 G ds$$

with n_i implying $n(\vec{k}_i)$

$$D^3 = n_1 n_3 (n_4 - n_2) + n_2 n_4 (n_3 - n_1)$$

where and

$$G = C(k_1, k_2, k_3, k_4) \left| \frac{\partial W}{\partial k_n} \right|^{-1} H(|\vec{k}_1 - \vec{k}_4| - |\vec{k}_1 - \vec{k}_3|)$$

The function C is the coupling coefficient. The function W is defined as the locus of points such that

$$W = \omega_1 + \omega_2 + \omega_3 + \omega_4 = 0$$

where ω_i is the radial frequency of the interacting waves.

The denominator in the Jacobian, k_n is the wavenumber vector normal to the interaction locus and H is the Heaviside function.

The form of equation 2 shows that interacting waves constitute a three-dimensional volume (Figure 1) imbedded within the original six-dimensional $\vec{k}_2 - \vec{k}_3 - \vec{k}_4$ interaction space, as was also shown in the work of Masuda (1980) who solved a transformed version of equation 2. Similar to any numerical integration method, the FBI (either the Webb-Resio-Tracy, WRT, method or Masuda's method) must integrate over a sufficient extent of the three-dimensional interaction space at a sufficient resolution to achieve a given degree of accuracy.

For many years researchers have sought a simplification to equation 2 which could either reduce the number of dimensions required in the integration or reduce the extent of the volume that had to be included within the integral while retain a prescribed accuracy, with very little success. The reason for this lack of success can be found in the fact that although both the density term in the integral, D^3, and the geometric phase-space term in the integral, G, can vary by several orders of magnitude within fairly small regions of the interaction space, their variation is essentially independent. Since the two terms are multiplicative, the contributions to the total integral cannot be prescribed by consideration of only one of the two terms without consideration of the other. Thus, in some cases, the primary contributions to the rate of change at a particular spectral location can come from one region of the interaction space, while for other spectral shapes the primary contributions can come from another region of the interaction space.

The original argument in favor of the made by Hasselmann *et al.* (1985) was that contributions to S_{nl} term would be dominated by interacting triplets in which at least two of the components were located in the vicinity of the spectral peak. Whereas this is might be true for interactions in the vicinity of the spectral peak for relatively peaked wave spectra, it is not a good approximation for all spectral regions even in peaked spectra nor does it provide a good approximation in the spectral peak region for spectra which are not very peaked. Unfortunately, the precise formulation of the DIA is considerably more limiting than implied by the above assumption, since it essentially adds a delta function into the geometric phase space portion of equation 2 (the Phillips figure-8 form for the allowable interactions),

$$3.\quad G = C(k_1,k_2,k_3,k_4)\left|\frac{\partial W}{\partial k_n}\right|^{-1} H(|\vec{k}_1 - \vec{k}_4| - |\vec{k}_1 - \vec{k}_3|)\delta(\vec{k}_1 - \vec{k}_2)$$

Since the vector delta function removes two dimensions from the three dimensional integral, the interactions are now limited to a one-dimensional line with zero "interaction volume." The figure 8 shown in Figure 2 is the case where the value of s has been adjusted to the point where the delta function on $\vec{k}_1 - \vec{k}_2$ is satisfied.

Figures 3 and 4 show examples of how the Phillips' figure 8 constraint translates into conventional interactions considered in the WRT method for the specific cases of

$\vec{k}_1 = \vec{k}_2 = (1,0)$

with

$\vec{k}_3 = (1.14559, 0.101458)\ k_4 = (0.854481, -0.101458)$ (Figure 3) .

$\vec{k}_3 = (2.0, 0.33086)\ k_4 = (0.0, -0.33086)$ (Figure 4)

By now it should be recognized that the inclusion of the delta function on $\vec{k}_1 - \vec{k}_2$ is an extremely large "leap of faith" to make in an approximation to the full FBI, even if the interaction were computed over the entire figure-8 locus. Such an integral might require evaluations of the contributions to the integral at 30-50 points to maintain a high accuracy. Instead, the DIA further reduces the integral by choosing only 4 points (a quadruplet) along the figure 8, which further reduces the portion of the interaction space considered.

A logical question related to the selection of a single quadruplet along the figure-8 locus is "do the typical points selected represent locations of significant maxima in the coupling rates of interacting waves?" Figure 5 shows the coupling coefficients associated with the complete interacting loci associated with the figure 8 locus shown in Figure 2. In Figure 5, s is again defined such that the "0-value" coincides with the point satisfying the original quadruplets along the $\vec{k}_1 - \vec{k}_2$ delta function (the figure-8 locus). As can be seen here, the coupling coefficients are not particularly large for this location.

Even if all spectra had precisely the same shape, it would still not be possible to define a single constant which could be used to correctly scale the DIA at a fixed location to its corresponding FBI value for all frequencies and angles in a spectrum. This can be seen by noting that this empirical constant is definable as

4. $$Z_{DIA} = \frac{\iint 2\int D^3 G \delta(\vec{k}_1 - \vec{k}_2) ds d\vec{k}_3}{\iint 2\int D^3 G ds d\vec{k}_3}$$

Formally, the value of Z_{DIA} can be seen to be zero because the delta function reduces the integration volume to zero; however, in the sense that it is used by Hasselmann *et al.* (1985), it retains the phase areas associated with the specific interacting spectral components and does not vanish, introduces a dependence on spectral resolution into the value of Z_{DIA} . Although the coefficient Z_{DIA} defined here is not numerically equivalent to the empirical constant used in operational versions of the DIA, the points that are being made here should be equally valid for the empirical constant used in those versions of the DIA.

Given the scaling relationships inherent in the FBI (see for example Resio, 1987), if that the sampling locations (quadruplet locations) for the DIA retain the same geometric configuration throughout the spectrum, the effect of the $\vec{k}_1 - \vec{k}_2$ delta function on the geometric component's (the "G" function in equation 2) contribution to the ratio defining Z_{DIA} would be a constant independent of angle and frequency. However the action density triplets will be very different (the D^3 term in equation 4). Thus the value of the empirical constant required to equate the DIA to the FBI ratio at one frequency and angle will, in general, depend on the location within the spectrum. For example, in a small set of numerical tests, the values of Z_{DIA} for different points in the same JOSNWAP spectrum were found to vary by up to a factor of 10 and the difference between values at the same location within two JONSWAP spectra with different peakednesses was vary by a factor of 8. The fact that a single empirical constant is used in operational versions of the DIA produces results which distort the resulting nonlinear source terms obtained via the DIA, as shown in Resio and Perrie (2008) and Perrie and Resio (2009).

The basic discrepancy described here is that the value of Z_{DIA} varies as a function of location within the spectrum. Optimizations which fit an overall rms error term over the spectrum are basically attempting to find the value of Z_{DIA} which works best over the entire spectrum. This will emphasize the behavior of the approximation in the region where S_{nl} is highest, the spectral peak region, and may provide a poor approximation elsewhere.

It is unclear how much advantage, if any, is gained by including additional points around the figure-8 locus (as in the MDIA) and by relaxing the $\vec{k}_1 - \vec{k}_2$ delta-function constraint (the EDIA), since this does not remove the basis of the discrepancy created by global optimization noted here. In fact, any discrete approximation which uses a globally optimized single empirical constant to convert the local discretized approximation to an FBI estimate will distort the results. Furthermore, the use of a small subset of points from the complete three-dimensional interaction space to approximate the FBI will always lead to results which are substantially affected by the shape of the spectrum. Thus, as the spectral shape changes so does the location of the optimal sampling points and the (spectral-location dependent) values of the coefficients which can be used to equate the discretized estimates to the FBI estimates.

3. Analysis of DIA performance for parametric and observed spectra

Since a recent publication by two of the co-authors here have presented comparisons of the DIA to the FBI, we will use results from those papers (Resio and Perrie, 2008; and Perrie and Resio, 2009) rather than a set of new comparisons to examine the performance of the DIA in light of the theoretical discussion presented above.

Figure 6 from Resio and Perrie (2008) shows a comparison of the DIA to the FBI for some simple parametric spectra, JONSWAP spectra with a spectral peak frequency, f_p, of 0.1 hz, a value of the equilibrium constant, α, equal to 0.01, a value of the peakedness spreading parameters, σ_a and σ_b, equal to 0.07 and 0.09 respectively, and the peakedness parameter, γ, taking on three values (1.0, characteristic of a fully developed sea; 3.3 characteristic of a fetch limited wave spectrum; and 7.0 characteristic of a very narrow swell spectrum). As can be seen here, the DIA produces a prediction that varies systematically with respect to the FBI. It overpredicts values of S_{nl} (by almost a factor of 3) near the spectral peak for the low peakedness case. It was tuned to approximate the near-peak values of S_{nl} for the fetch-growth case, but still produces very poor results in other region of the spectrum as expected from the above discussion. And, it underpredicts the values of S_{nl} (by a factor of over 2) near the spectral peak region for the high peakedness case. There is really no surprise in these results, as the peakedness increases, the dominance of the near-peak interactions increases; thus, the standard empirical coefficient used to relate the DIA to the FBI was tuned to match the medium peakedness case. For lower peakedness cases, it will overestimate the portion of the contribution to the total integral that comes from this region. For higher peakedness cases, it will underestimate the near-peak contribution to the total integral. This is a problem that could perhaps be overcome by tuning the DIA coefficient to vary as a function of peakedness. Similarly, the consistently poor performance of the DIA in regions other than the spectral peak would require tuning for each angle and frequency separately. Of course, such a tuning would be specific to a JONSWAP spectrum with a fixed angular distribution.

Figures 7 and 8 from Perrie and Resio (2009) show the performance of the DIA for the case of observed spectra at a site about 5 km offshore from the Field Research Facility in Duck, North Carolina. As can be seen here, the DIA's estimates are incredibly erratic. This is due to its reliance on a very small number of discretized interactions as the basis for its approximation to S_{nl} out of all of the interactions which actually contribute to S_{nl}.

4. Desired attributed for an approximation to S_{nl}

As one might recognize from the analyses presented here, approximations to the FBI are very difficult to formulate to be both efficient and accurate. This raises a bit of a quandary, since convincing evidence from many independent studies have demonstrated the important role that these interactions play in the evolution of wave spectra. A relevant question that has never really been addressed within the wave modeling community is whether or not it better to use an inaccurate representation for S_{nl} that retains the same number of degrees of freedom as contained within the directional spectrum or is it better

to have a representation for S_{nl} that is accurate but lacks the same number of degrees of freedom as contained in the directional spectrum?

To begin to answer this question, it is important to adopt some clear metrics for the representation of S_{nl} within wave models. The original argument for including all of the degrees of freedom with the representation for S_{nl} was to allow this term to function in complex situations. For a relatively simple situation, such as fetch-limited wave growth under a constant wind field, a simple parametric representation should be able to represent S_{nl} quite reasonably. For more complex scenarios, such as multiple wave trains approaching a coast, waves within hurricanes, or waves in other complex fetch/wind-field situations, it is extremely unlikely that a simple parametric-based representation of S_{nl} will provide reasonable accuracy. An example of this can be found in Figure 9 from Perrie and Resio (2009) where it is shown that even in a situation where the directionally integrated spectrum is very closely approximated by the spectrum used to derived the parameterized results for S_{nl}, the effects of different angular distributions can still produce large deviations between parametric estimates of S_{nl} and the FBI results.

Some metrics for an approximation to S_{nl} which might help us determine the best way ahead for wave modeling are listed below.

1. It should conserve the constants of motion (action, energy, and momentum); otherwise, the need for spurious additional source terms will undoubtedly arise.
2. It should produce the correct fluxes of action, energy and momentum through the spectrum in order to allow the spectral shapes to evolve in a proper fashion. The DIA does not produce a constant action flux for an f^{-4} (or $k^{-5/2}$) spectrum. This will produce a different partitioning of all the fluxes within a spectrum making it difficult to simulate the wave generation/dissipation process correctly.
3. It should force a response to a perturbation in the spectral densities that is quantitatively close to the FBI solution.
4. It should produce estimates of S_{nl} with deviations from the FBI that consistently deviate less than some small percentage, perhaps 15%, or so.
5. It should be accurately adaptable to coastal water depths. The scaling used in current Third-Generation models use a scaling that is too crude to capture important features of spectral evolution in waves propagating toward a coast.
6. The approximation in combination with reasonably posed additional source terms should be capable of producing rates of change of dimensionless energy and peak frequency across a fetch up to a condition of full development that agree with previous field studies.
7. The approximations in combination with reasonably posed additional source terms should be capable of producing directionally integrated spectra and directional distributions of energy which are consistent with previous field studies.

5. Discussion and Conclusions

The purpose of this paper is to provoke some recognition and discussion of categorical problems with the DIA's approach to estimating S_{nl} in wave models. It seems likely that most of the criticisms raised here will apply to some extent to all of the attempts to extend the DIA via the incorporation of additional quadruplets, either on the Phillips' figure-8 locus (MDIA) or off of this locus (EDIA). Although it has been widely accepted that nonlinear interactions play a dominant role in the wave generation process in all depths of water and in the wave transformation process in waves approach a coast, the present approximations appear to be too inaccurate to provide a realistic representation of S_{nl} in offshore and coastal areas.

Some conclusions based on the analyses here are as follows:

1. A small number of points selected from the entire set of potential interaction combinations does not seem able to provide a consistent, accurate estimate of S_{nl}, even for simple, parametric spectral shapes.
2. A single coefficient cannot be used to relate the values of S_{nl} to the FBI estimate at different locations in the spectrum, as is currently done in operations version of the DIA.
3. The impact of using discrete interactions is to introduce erratic errors into estimates of S_{nl} when the spectra contain perturbations from a smooth form.

Based of these conclusions, it seems necessary to improve the estimates of S_{nl} in a fashion consistent with the metrics suggested in this paper before a lot more effort is expended on adding additional nuances into existing source terms in attempts to coerce them into producing more accurate wave predictions in operational models.

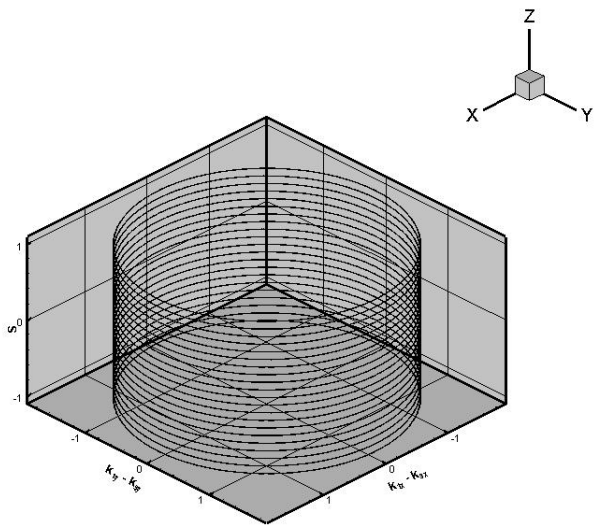

Figure 1. Graphical representation of the interaction volume considered within the Full Boltzmann Integral (FBI), with two dimensions introduced by the variation in $\vec{k}_3$ around $\vec{k}_1$ and one dimension introduced by the distance along the "s locus." All of the volume inside the cylinder would be considered in a WRT integration.

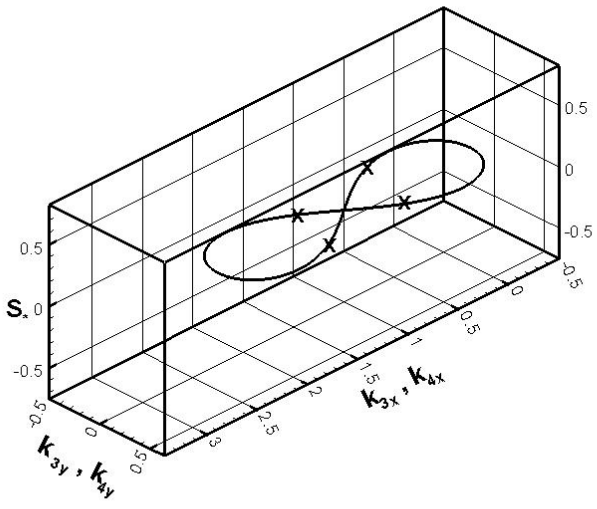

Figure 2. Location of the Phillips' figure-8 curve within the general 3-dimensional volume for wave-wave interactions. The "x's" in this figure are indicative of the location of a fixed quadruplet. Thus, in the DIA only 4 "points" are retained from the actual 3-dimensional interaction volume.

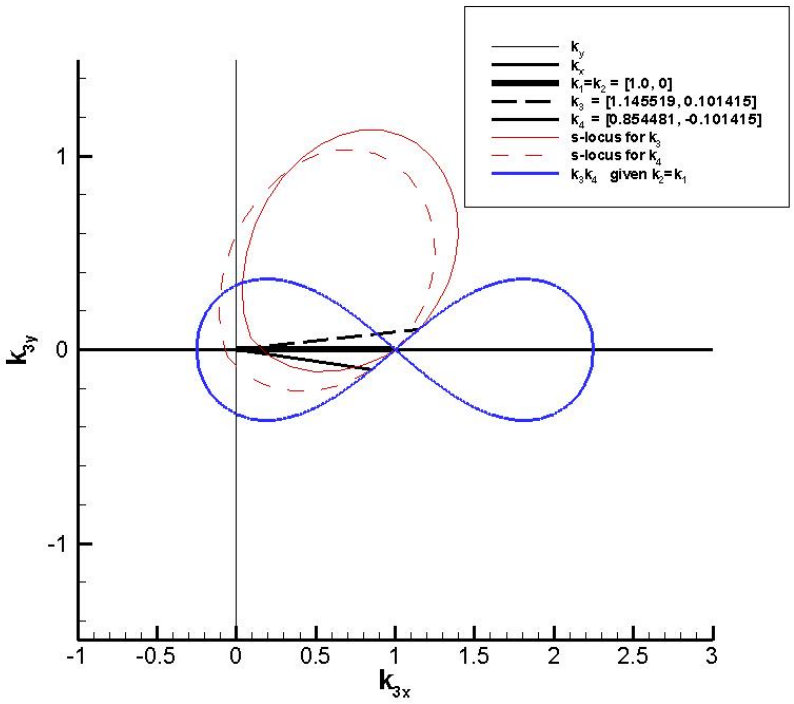

Figure 3. An example of the relationship between the Phillips figure-8 interaction locus and interactions considered within the WRT integration method.

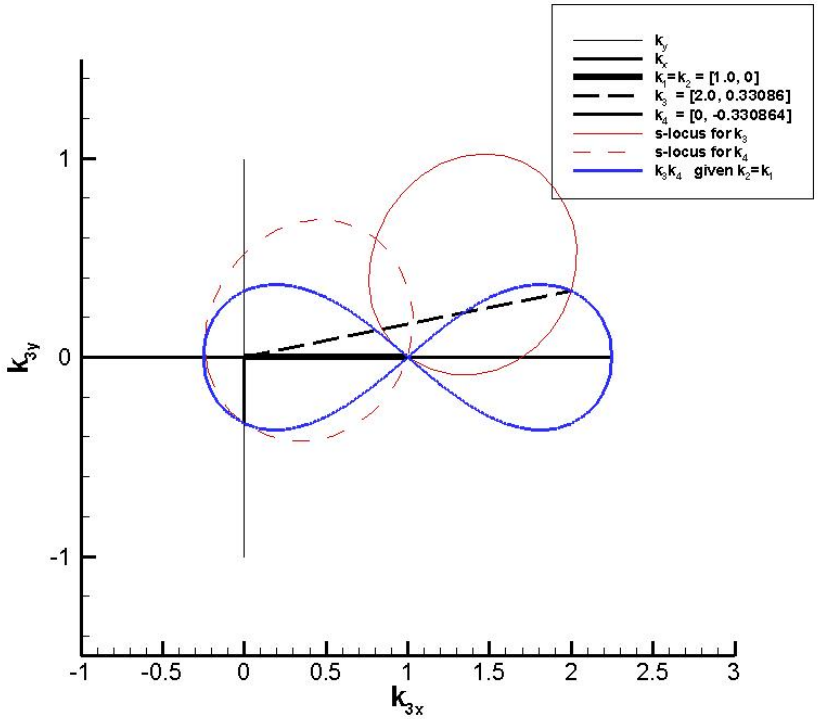

Figure 4. An example of the relationship between the Phillips figure-8 interaction locus and interactions considered within the WRT integration method.

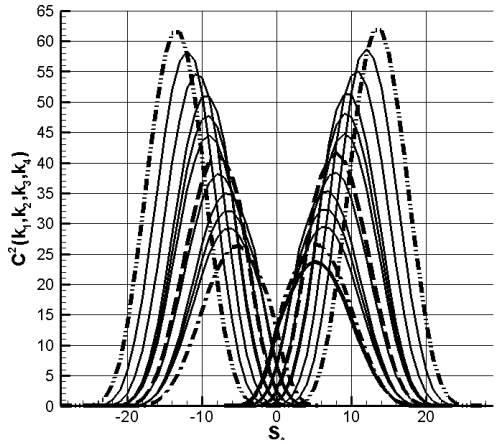

Figure 5. Variation in the coupling coefficient around the Phillips figure-8 locus. The definition of s=0 is such that it is coincident with the locations of the quadruplets in the original version of the DIA.

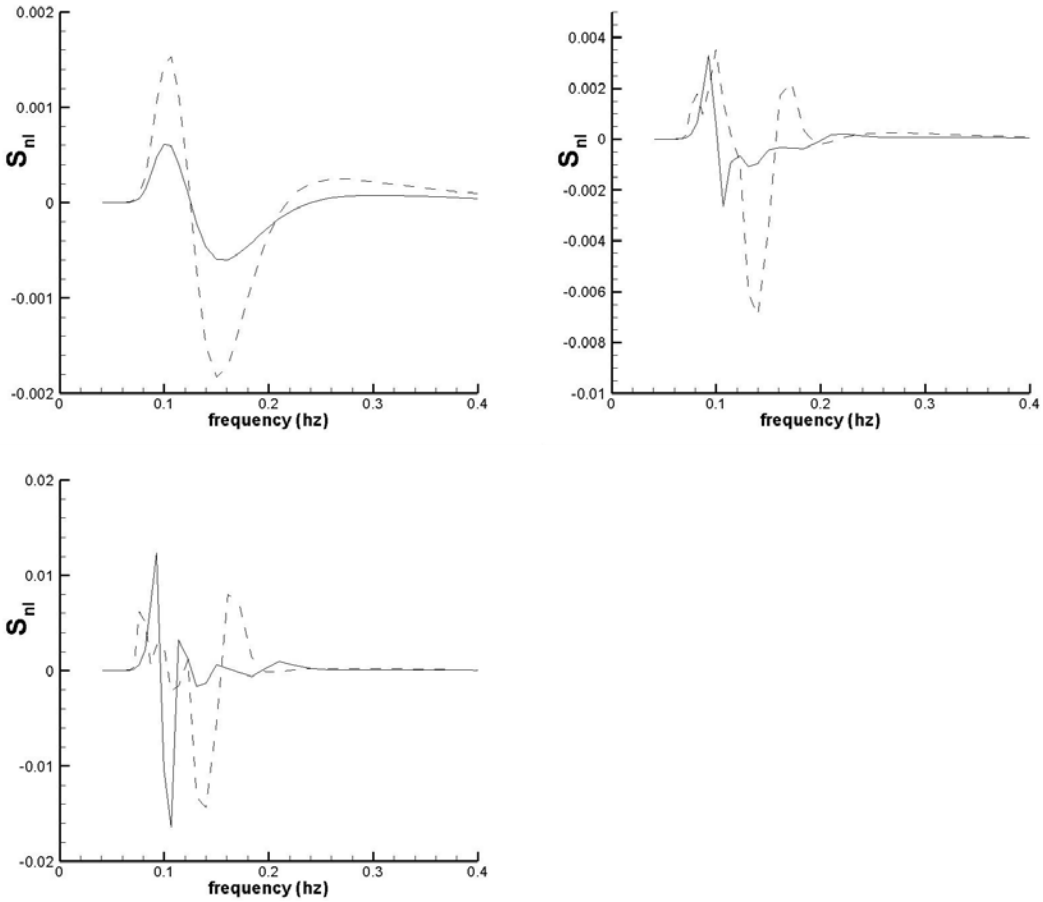

Figure 6. Comparison of DIA to FBI for JONSWAP spectra with 3 different peakedness values from Resio and Perrie (2008).

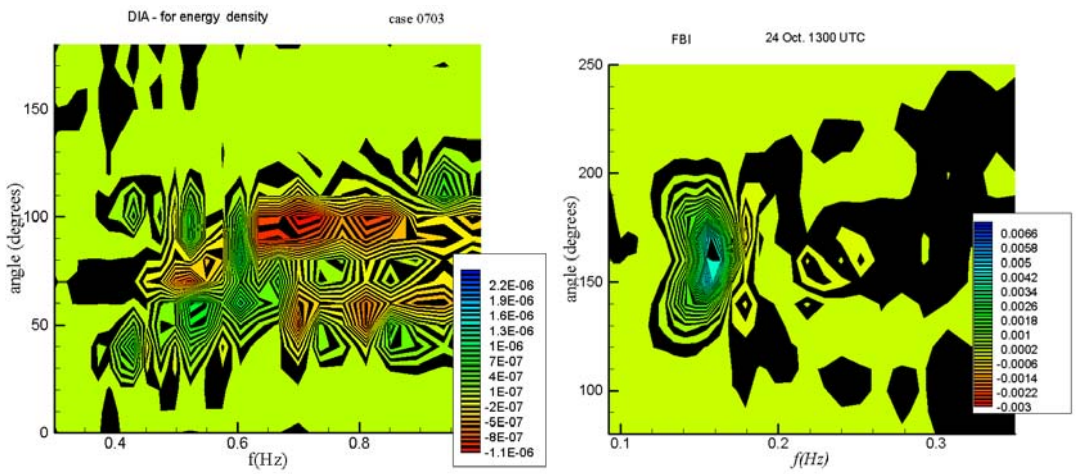

Figure 7. Comparison of an estimate of S_{nl} from the DIA compared to the FBI solution for a Currituck Sound case (Perrie and Resio, 2009).

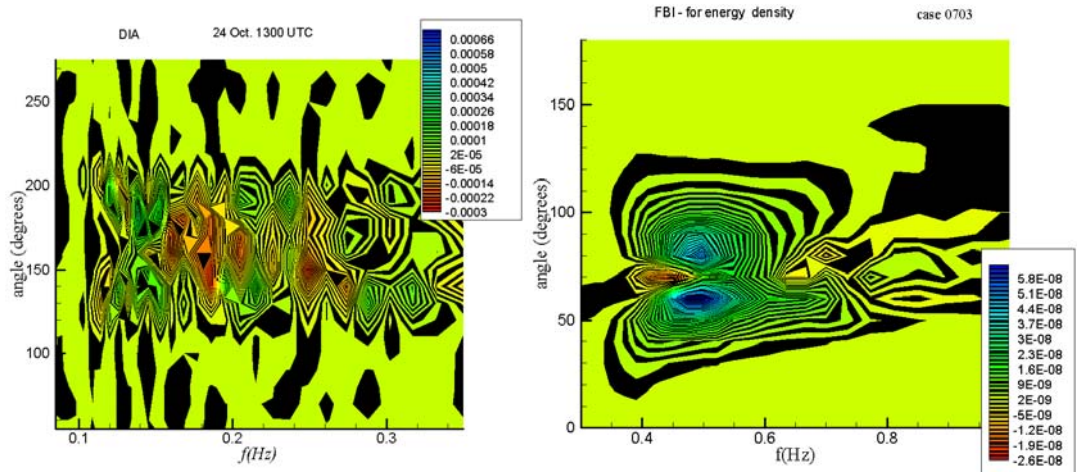

Figure 8. Comparison of an estimate of S_{nl} from the DIA compared to the FBI solution for a spectrum taken from the offshore waverider at the Field Research Facility in Duck, North Carolina (Perrie and Resio, 2009).