On Domination of Nonlinear Wave Interaction in the Energy Balance of Wind-Driven Sea

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1 Introduction

In the presented talk we discuss some theoretical aspects of the physics of wind-driven sea. On our opinion, some important questions of this theory are not clarified enough and must be elucidated. This clarification is necessary to provide an adequate comparison of the theory and the experiment, otherwise costly and laborious field and laboratory measurements could not be properly interpreted and understood.

The first question is about the correct definition of wave action $N_k(t)$, which obeys the Hasselmann kinetic equation

$$\frac{dN}{dt} = S_{nl} + S_{in} + S_{dis}, \tag{1.1}$$

augmented by the source and the dissipation terms. How to find the current action spectrum $N_k(t)$ from experimental data? What is measured in the best experiments, is the space-time spectrum

$$Q_{k\omega} =< |\eta_{k\omega}|^2 > . \tag{1.2}$$

Here $\eta_{k\omega}$ is the Fourier transform of the surface elevation. The most advanced definition of wave action, used in many research papers (see, for example [1,2]), is the following:

$$N_k = \frac{2}{\omega_k} \int_0^\infty Q_{k\omega}\, d\omega. \tag{1.3}$$

Formula (1.3) is certainly correct for waves of very small amplitude in the limit $\mu \to 0$, where μ is a characteristic average steepness of the surface. At a finite steepness, it can be treated as the first term in expansion

$$N_k = N_0(k) + \mu^2 N_1(k) + \cdots . \tag{1.4}$$

Now $N_0(k)$ is given by (1.3), while $N_1(k)$ is the subject for determination. One can think that this question is not very important because even for the most steep young waves $\mu^2 \simeq 0.01$, and the accuracy of (1.3) looks good. However, our preliminary estimates show that the ratio $N_1(k)/N_0(k)$ is a fast growing function on k, thus for spectral tails the difference between N_k and $N_0(k)$ might be essential.

Now we formulate the inverse problem. Suppose we know N_k. How to find $Q_{k\omega}$?

In the linear approximation, at $\mu \to 0$, the answer is known:

$$Q_{k\omega} = \frac{\omega_k}{2} \left(N_k \, \delta(\omega - \omega_k) + N_{-k} \, \delta(\omega + \omega_{-k}) \right). \tag{1.5}$$

What happens if μ is finite? In the neighborhood of $\omega = \omega_k$ we should perform replacement

$$\delta(\omega - \omega_k) \to \frac{1}{\pi} \frac{\Gamma_k}{(\omega - \tilde{\omega}_k)^2 + \Gamma_k^2}, \tag{1.6}$$

where $\tilde{\omega}_k = \omega_k + \mu^2 \omega_{1k} + \cdots$ is renormalized frequency and $\Gamma_k \simeq \mu^4 \tilde{\Gamma}_k + \cdots$ is effective dissipation due to four-wave processes. As far as μ^2 is small, one can think that both shifting of ω_k and blurring of δ-function are weak effects. However, the quotients ω_{1k}/ω_k and $\tilde{\Gamma}_k/\omega_k$ are growing functions on k, thus for $k \gg k_p$ (k_p is the wave number of spectral peak) derivation from simple formula (1.5) could be essential. There is one more important effect. In the real sea all waves could be separated in two classes: "resonant waves" with $\omega \sim \omega_k$ and "slave harmonics" caused by quadratic nonlinearity of primitive dynamic equations. The slave waves do not obey dispersion relations, as a result their frequency spectrum for the given k is a broad function, not concentrated at $\omega \simeq \omega_k$.

Accurate determination of $N_1(k)$ at given $Q_{k\omega}$ and $Q_{k\omega}$ at given $N(k)$ is possible but it is technically cumbersome problem. In Chapters 2, 3 we are taking first but important steps to their solution. In Chapter 4 we study axial asymmetric solutions of equation

$$S_{nl} = 0, \tag{1.7}$$

that is known since 1966 ([3], see also [4, 5]). This equation has exactly two powerlike solutions:

$$N_1(k) = c_p \left(\frac{P}{g^2} \right)^{1/3} \frac{1}{k^4}, \tag{1.8}$$

$$N_2(k) = c_q \left(\frac{Q}{g^{3/2}} \right)^{1/2} \frac{1}{k^{23/6}}. \tag{1.9}$$

Solution (1.8) is known as Zakharov-Filonenko spectrum [4]. Here P is the flux of energy from small wave numbers and Q is the flux of wave action from high wave numbers. Kolmogorov constants c_p and c_q were not known but now they are calculated:

$$c_p = 0.219, \qquad c_q = 0.227. \tag{1.10}$$

General isotropic solutions of Eq. (1.7) depend on two constants P and Q. In Chapter 5 we discuss the general anisotropic solution of this equation. We show that the solution is defined by one arbitrary constant, the flux of wave action from high wave numbers, and one arbitrary function on angle. In the axially symmetric case this function degenerates to the constant P. The general anisotropic solution of (1.7) describes angular spreading of spectrum growing with frequency. The last Chapter 6 is the most important from the practical view-point. We discuss the balance equation in the universal domain $\omega \gg \omega_p$,

$$S_{nl} + S_{in} + S_{dis} = 0. \tag{1.11}$$

Apparently in some domain on k-plane $S_{in} + S_{dis} > 0$. Suppose that $S_{in} = \gamma(k)\, N_k$. We notice that S_{nl} can be presented in the form

$$S_{nl} = F_k - \Gamma_k\, N_k, \tag{1.12}$$

and the nonlinear wave interaction process is predominating if $\Gamma_k \gg \gamma_k$. We show that this condition is satisfied in majority of realistic cases, if the waves are not very young. It means that, as we claimed before, the nonlinear wave interaction is the dominating process in the wind-driven sea.

2 What is the wave action?

This is the widely used Hasselmann equation:

$$\frac{\partial N}{\partial t} + \frac{\partial \tilde{\omega}}{\partial \vec{k}} \frac{\partial N}{\partial \vec{r}} = S_{nl}, \tag{2.1}$$

$$\begin{aligned} S_{nl} = \pi g^2 \int |T_{kk_1,k_2k_3}|^2\, \delta(k + k_1 - k_2 - k_3)\, \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times \\ \times (N_{k_1} N_{k_2} N_{k_3} + N_k N_{k_2} N_{k_3} - N_k N_{k_1} N_{k_2} - N_k N_{k_1} N_{k_3}) dk_1 dk_2 dk_3. \end{aligned} \tag{2.2}$$

Here $\omega_k = \sqrt{g\, k\, \tanh kH}$, H is depth, $T_{kk_1k_2k_3} = T_{k_1kk_2k_3} = T_{k_2k_3kk_1} = T_{kk_1k_3k_2}$ are coupling coefficients, and

$$\tilde{\omega}(k) = \omega(k) + 2g \int T_{kk_1,kk_1}\, N_{k_1}\, dk_1 \tag{2.3}$$

is renormalized frequency.

As it was mentioned before, the nonlinear interaction term S_{nl} can be presented in the form

$$S_{nl} = F_k - \Gamma_k\, N_k, \tag{2.4}$$

where

$$F_k = \pi g^2 \int |T_{kk_1k_2k_3}|^2\, \delta(k+k_1-k_2-k_3)\, \delta(\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3})\, N_{k_1} N_{k_2} N_{k_3}\, dk_1 dk_2 dk_3 \tag{2.5}$$

and Γ_k, the dissipation rate due to the presence of four-wave processes, is the following:

$$G_k = \pi g^2 \int |T_{kk_1,k_2k_3}|^2\, \delta(k + k_1 - k_2 - k_3)\, \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times \tag{2.6}$$

$$\times (N_{k_1}N_{k_2} + N_{k_1}N_{k_3} - N_{k_2}N_{k_3})\, dk_1 dk_2 dk_3. \tag{2.7}$$

One can say that in the real nonlinear sea the dispersion relation $\omega = \omega_k$ is renormalized and becomes a complex function

$$\omega_k \to \tilde{\omega}_k + \frac{1}{2}\, i\Gamma_k. \tag{2.8}$$

Eq. (2.1), (2.2) are written for the wave action spectrum $N_k(\vec{r}, t)$. What is the exact definition for the wave action? How $N_k(\vec{r}, t)$ can be expressed through the observable measurable quantities? These are not that simple questions.

Making a snapshot of the surface from two points one can get its stereoscopic image and restore the shape of elevation $\eta(\vec{r})$. If we perform nonsymmetric Fourier transform and define

$$\eta_k = \frac{1}{(2\pi)^2} \int \eta(\vec{r})\, e^{-ikr}\, d\vec{r}, \tag{2.9}$$

we can introduce the spatial spectrum

$$Q_k = < |\eta_k|^2 > . \tag{2.10}$$

Making a seria of snapshots in consequent moments of time one can restore the full space-time spectrum

$$Q_{k\omega} = < |\eta_{k\omega}|^2 > . \tag{2.11}$$

Apparently,

$$Q_k = \int_{-\infty}^{\infty} Q_{k\omega}\, d\omega. \tag{2.12}$$

What is the wave action N_k? In some articles and monographs we can find the following definition:

$$N_k = \frac{Q_k}{\omega_k}. \tag{2.13}$$

This is just a widely spread carelessness. Spectrum Q_k is an even function, $Q_{-k} = Q_k$, while N_k certainly does not obey this restriction. One can present the spatial spectrum in the form

$$Q_k = \frac{\omega_k}{2}(n_k + n_{-k}), \tag{2.14}$$

where n_k is the wave action. We deliberately denoted it by low-case letter, because n_k and N_k are *different* wave actions.

The wave field consists of "resonant" and "slave" harmonics. The resonant harmonic with wave vector $\vec{k}$ has a frequency close to the renormalized frequency $\tilde{\omega}_k$. The most strong slave harmonics appear as a result of interaction of two resonant harmonics. Suppose they have wave vectors $\vec{k}_1, \vec{k}_2$. In the first order of nonlinearity they generate four slave harmonics with wave vectors $\vec{p}_1, \vec{p}_2, -\vec{p}_1, -\vec{p}_2$ and frequencies $\Omega_1, \Omega_2, -\Omega_1, -\Omega_2$. Here $\vec{p}_1 = \vec{k}_1 - \vec{k}_2$, $\vec{p}_2 = \vec{k}_1 + \vec{k}_2$, and $\Omega_1 = \omega_1 - \omega_2$, $\Omega_2 = \omega_1 + \omega_2$. There is no any definite relation between the wave vector and the frequency for slave harmonics.

Returning to the wave action, let us explain now the difference between n_k and N_k. N_k is the "refined" wave action that includes resonant harmonics and slave harmonics of higher order only and n_k is the "total" wave action that includes both resonant and all slave harmonics. Apparently, $n_k > N_k$ and is directly connected with experimentally measurable spatial spectrum by relation (2.14). But n_k *does not obey* the Hasselmann equation. On the contrary, the "purified" wave action N_k *in principle* cannot be measured in any kind of experiment. But exactly this sort of wave action satisfies the Hasselmann equation. As a result, all operational models solve the Hasselmann equation augmented with additional terms: S_{in}, the input from wind, and S_{dis}, the dissipation due to wave breaking. Hence the operational models do predict N_k. At the same time, experimentalists can measure the n_k only.

On the first glance we see serious discrepancy, however nobody pays any attention. Why this happens?

To give an answer we should estimate the relative difference between n_k and N_k. Let us denote

$$\alpha(k) = \frac{n_k - N_k}{n_k}. \tag{2.15}$$

In a typical observed spectrum of wind-driven sea we should separate spectral area near to the peak frequency $\omega \sim \omega_p$ and the tail $\omega \gg \omega_p$. In the energy capacitive spectral band close to ω_p, α is small:

$$\alpha \sim \mu^2.$$

The characteristic steepness μ is defined as

$$\mu^2 \simeq \frac{\omega_p^4}{g^2}\,\sigma^2,$$

where σ is the total energy of waves. Even for young waves $\mu^2 \leq 0.01$, thus the relative difference between n and N for deep water is not more than one percent and can easily be neglected. However, $\alpha(k)$ is a fast growing function on k. An accurate estimate of dependance α on frequency at $\omega \geq \omega_p$ is not a subject for current research. The article on this topic will be presented for publication soon, however our preliminary results show that this dependance is very fast growing:

$$\alpha \simeq \mu^2 \left(\frac{\omega}{\omega_p}\right)^3. \tag{2.16}$$

As it was mentioned above, in the area $\omega \sim \omega_p$ one can neglect the difference between n_k and N_k. In this area we can replace Eq. (2.10) by

$$Q_k = \frac{\omega_k}{2}(N_k + N_{-k}). \tag{2.17}$$

There is essential difference between (2.14) and (2.17). Because $n_k > 0$ at any k, wave vectors of slave harmonics cover all k-plane, thus determination of n_k from Q_k is impossible in principle. On the contrary, in many practical cases N_k is nonzero only inside the bounded domain G on the k-plane. At the same time $N_{-k} \neq 0$ inside the domain $\tilde{G}$ only, which is radially symmetric to G. In other words, if vector $\vec{k}$ belongs to G, vector $-\vec{k}$ belongs to $\tilde{G}$. Suppose that G and $\tilde{G}$ have no intersection. In this case in the domain G we have $N_k = 2Q_k/\omega_k$. In spite of presence of factor 2 in (2.14) the integral identity $\int Q_k\, dk = \int \omega_k\, N_k\, dk$ is the same as we would have used the naive and blatantly incorrect formula (2.13).

In some important cases domains G and $\tilde{G}$ have intersection. In this case we face ambiguity in determination of N_k from (2.17). To overcome this ambiguity one should use the space-time spectrum $Q_{k,\omega}$ and define

$$n_k = \frac{2}{\omega_k} \int_0^\infty Q(k,\omega)\, d\omega. \tag{2.18}$$

The equivalent formula is presented in the monograph of Monin and Krasitsky [1] printed in Russia in 1985. It was also used by Rosental et al [2] approximately in the same time. In this case again

$$\int \omega_k\, n_k\, dk = \int_{-\infty}^{\infty} Q(k,\omega)\, d\omega dk. \tag{2.19}$$

Let us notice that formulae (2.14), (2.18) account slave harmonics and can be used with comparison of spectral tails obtained from the experiment and from solution of Hasselmann equation, both numerical and analytical, with caution. They work up to accuracy of μ^2 in the neighborhood of spectral peak, but can lead to essential errors in area of spectral tails. Preliminary estimate for accuracy of expression (2.18) will be done in the next Chapter.

3 How to separate resonant and slave harmonics?

To make the accurate separation of resonant and slave harmonics and to find an explicit formula that connects $Q(k,\omega)$ and N_k, one should use Hamiltonian formalism and implement the canonical transformation, excluding cubic terms in the Hamiltonian. This is a cumbersome mathematical procedure. In this Chapter we will demonstrate how it could be done in the most economic way.

We study the weakly nonlinear waves on the surface of an ideal fluid of infinite depth in an infinite basin. The vertical coordinate is

$$-H < z < \eta(r,t), \qquad r = (x,y), \tag{3.1}$$

the fluid is incompressible, H is the depth of fluid,

$$div\, V = 0, \tag{3.2}$$

and velocity V is a potential field

$$V = \nabla\,\Phi, \tag{3.3}$$

where potential Φ satisfies the Laplace equation

$$\Delta\,\Phi = 0 \tag{3.4}$$

under boundary conditions

$$\Phi|_{z=\eta} = \Psi(r,t), \qquad \Phi_z|_{z=-\infty} = 0. \tag{3.5}$$

The total energy of the fluid, $H = T + U$, has the following terms:

$$T \;=\; \frac{1}{2}\int d\vec{r}\int_{-\infty}^{\eta}(\nabla\,\Phi)^2\,dz = \frac{1}{2}\int \Psi\,\Phi_n\,dS, \tag{3.6}$$

$$U \;=\; \frac{1}{2}\,g\int \eta^2\,d\vec{r}. \tag{3.7}$$

The Dirichlet-Neumann boundary problem (3.4), (3.5) is uniquely resolved; thus the flow is defined by fixation of η and Ψ. This pair of variables is canonical; thus evolution equations for η, Ψ take the form [6]:

$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\Psi}, \qquad \frac{\partial\Psi}{\partial t} = -\frac{\delta H}{\delta\eta}. \tag{3.8}$$

After non-symmetric Fourier transform,

$$\Psi(r) = \int \Psi(k)\,e^{ikr}\,dk, \qquad \Psi(k) = \frac{1}{(2\pi)^2}\int \Psi(r)\,e^{-ikr}\,dr, \tag{3.9}$$

equation (3.8) reads:

$$\frac{\partial\eta}{\partial t} = \frac{\delta\tilde{H}}{\delta\Psi_k^*}, \qquad \frac{\partial\Psi}{\partial t} = -\frac{\delta\tilde{H}}{\delta\eta_k^*}, \tag{3.10}$$

$$\tilde{H} = \frac{1}{4\pi^2}\,H = H_0 + H_1 + H_2 + \cdots \tag{3.11}$$

In [7-9] was shown that Hamiltonian $\tilde{H}$ can be expanded in Taylor series in powers of $k\eta_k$:

$$\begin{aligned}
H_0 &= \frac{1}{2}\int\left\{A_k|\Psi_k|^2 + g\,|\eta_k|^2\right\}dk, \quad A_k = k\tan kH \\
H_1 &= \frac{1}{2}\int L^{(1)}(k_1,k_2)\Psi_{k_1}\Psi_{k_2}\eta_{k_3}\,\delta(\vec{k_1}+\vec{k_2}+\vec{k_3})\,dk_1\,dk_2\,dk_3 \\
H_2 &= \frac{1}{2}\int L^{(2)}(k_1,k_2,k_3,k_4)\Psi_{k_1}\Psi_{k_2}\eta_{k_3}\eta_{k_4}\,\delta(k_1+k_2+k_3+k_4)\,dk_1 dk_2\eta_{k_3}\eta_{k_4}
\end{aligned} \tag{3.12}$$

Here

$$L^{(1)}(k_1,k_2) = -(k_1,k_2) - A_{k_1}\,A_{k_2} \tag{3.13}$$

$$L^{(2)}(k_1,k_2,k_3,k_4) = \frac{1}{2}\,(k_1^2A_2 + k_2^2\,A_1) + \frac{1}{4}A_1A_2(A_{1+3} + A_{2+4} + A_{1+4} + A_{2+3})$$

Now we can introduce normal variables a_k:

$$\begin{aligned}\eta_k &= \frac{1}{\sqrt{2}}\left(\frac{A_k}{g}\right)^{1/4}(a_k + a^*_{-k}) \\ \Psi_k &= \frac{i}{\sqrt{2}}\left(\frac{g}{A_k}\right)^{1/4}(a_k - a^*_{-k}) \end{aligned} \tag{3.14}$$

Normal variables obey the following Hamiltonian equations:

$$\frac{\partial a_k}{\partial t} + i\frac{\delta H}{\delta a^*_k} = 0 \tag{3.15}$$

All terms in the expansion of Hamiltonian (3.11) must be expressed in terms of a_k:

$$\begin{aligned} H_0 &= \int \omega_k |a_k|^2 dk \\ H_1 &= \frac{1}{2}\int V^{(1,2)}_{kk_ak_2}(a_k a^*_{k_1} a^*_{k_2} + a^*_k a_{k_1} a_{k_2})\delta(k - k_1 - k_2)dkdk_1dk_2 + \\ &+ \frac{1}{6}\int V^{(0,3)}_{kk_ak_2}(a_k a_{k_1} a_{k_2} + a^*_k a^*_{k_1} a^*_{k_2})\delta(k + k_1 + k_2)dkdk_1dk_2 \end{aligned} \tag{3.16}$$

$$V^{(1,2)}_{kk_1k_2} = \frac{g^{1/4}}{2\sqrt{2}}\left\{\left(\frac{A_k}{A_{k_1}A_{k_2}}\right)^{1/4} L^{(1)}(k_1,k_2) - \left(\frac{A_{k_1}}{A_kA_{k_2}}\right)^{1/4} L^{(1)}(-k,k_1) - \left(\frac{A_{k_2}}{A_kA_{k_1}}\right)^{1/4} L^{(1)}(-k,k_2)\right\} \tag{3.17}$$

$$V^{(0,3)}_{kk_1k_2} = \frac{g^{1/4}}{2\sqrt{2}}\left\{\left(\frac{A_k}{A_{k_1}A_{k_2}}\right)^{1/4} L^{(1)}(k_1,k_2) + \left(\frac{A_{k_1}}{A_kA_{k_2}}\right)^{1/4} L^{(1)}(k,k_1) + \left(\frac{A_{k_2}}{A_kA_{k_1}}\right)^{1/4} L^{(1)}(k,k_2)\right\} \tag{3.18}$$

Now we can define the "total" or rough action:

$$n_k\,\delta(k - k') = g < a_k\, a^*_{k'} > . \tag{3.19}$$

It is clear that fundamental relation (2.14) is satisfied. Then, we perform the Fourier transform in time

$$a_{k\omega} = \frac{1}{2\pi}\int a(k,t)e^{-i\omega t}dt \tag{3.20}$$

and introduce

$$n_{k\omega}\,\delta(k - k')\,\delta(\omega - \omega') = g < a_{k\omega}\, a^*_{k',\omega'} > . \tag{3.21}$$

The space-time spectrum of elevation is simply

$$Q_{k,\omega} = \frac{\omega_k}{2}(n_{k,\omega} + n_{-k,-\omega}) \tag{3.22}$$

To separate resonant and slave harmonics we must perform a canonical transformation to new variables, excluding cubic terms in the Hamiltonian. This is a standard procedure known in celestial dynamics down to nineteenth century. However in our case this procedure is rather cumbersome. It was first done by Krasitski [9]. He found transformation of initial canonical variables a_k to new canonical variables b_k, which contain first order slave harmonics only. Variables a_k are presented by infinite series in new variables b_k:

$$a_k = b_k + a_k^{(1)} + a_k^{(2)} + a_k^{(3)}. \tag{3.23}$$

He calculated first two terms in this expansion and found the following expressions:

$$\begin{aligned}
a_k^{(1)} &= \int \Gamma^{(1)}(\vec{k},\vec{k}_1,\vec{k}_2)\, b_{k_1}\, b_{k_2}\, \delta(\vec{k}-\vec{k}_1-\vec{k}_2)\, dk_1\, dk_2 \\
&- 2\int \Gamma^{(1)}(\vec{k}_2,\vec{k},\vec{k}_1)\, b^*_{k_1}\, b_{k_2}\, \delta(\vec{k}+\vec{k}_1-\vec{k}_2)\, dk_1\, dk_2 \\
&+ \int \Gamma^{(2)}(\vec{k},\vec{k}_1,\vec{k}_2)\, b^*_{k_1}\, b^*_{k_2}\, \delta(\vec{k}+\vec{k}_1+\vec{k}_2)\, dk_1\, dk_2 \\
a_k^{(2)} &= \int B(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3)\, b^*_{k_1}\, b_{k_2}\, b_{k_3}\, \delta(\vec{k}+\vec{k}_1-\vec{k}_2-\vec{k}_3)\, dk_1\, dk_2\, dk_3 + \cdots
\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}
\Gamma^{(1)}(\vec{k},\vec{k}_1,\vec{k}_2) &= -\frac{1}{2}\frac{V^{(1,2)}(\vec{k},\vec{k}_1,\vec{k}_2)}{(\omega_k-\omega_{k_1}-\omega_{k_2})} \\
\Gamma^{(2)}(\vec{k},\vec{k}_1,\vec{k}_2) &= -\frac{1}{2}\frac{V^{(0,3)}(\vec{k},\vec{k}_1,\vec{k}_2)}{(\omega_k+\omega_{k_1}+\omega_{k_2})}
\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
&B(\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3) = \\
&\Gamma^{(1)}(\vec{k}_1,\vec{k}_2,\vec{k}_1-\vec{k}_2)\,\Gamma^{(1)}(\vec{k}_3,\vec{k},\vec{k}_3-\vec{k}) + \Gamma^{(1)}(\vec{k}_1,\vec{k}_3,\vec{k}_1-\vec{k}_3)\,\Gamma^{(1)}(\vec{k}_2,\vec{k},\vec{k}_2-\vec{k}) \\
&-\Gamma^{(1)}(\vec{k},\vec{k}_2,\vec{k}-\vec{k}_2)\,\Gamma^{(1)}(\vec{k}_3,\vec{k}_1,\vec{k}_3-\vec{k}_1) - \Gamma^{(1)}(\vec{k}_1,\vec{k}_3,\vec{k}_1-\vec{k}_3)\,\Gamma^{(1)}(\vec{k}_2,\vec{k}_1,\vec{k}_2-\vec{k}_1) \\
&-\Gamma^{(1)}(\vec{k}+\vec{k}_1,\vec{k},\vec{k}_1)\,\Gamma^{(1)}(\vec{k}_2+\vec{k}_3,\vec{k}_2,\vec{k}_3) + \Gamma^{(2)}(-\vec{k}-\vec{k}_1,\vec{k},\vec{k}_1)\,\Gamma^{(2)}(-\vec{k}_2-\vec{k}_3,\vec{k}_2,\vec{k}_3)
\end{aligned} \tag{3.26}$$

On our opinion, Krasitski used a rather long way for calculation of terms in expansion (3.23). He directly checked the validity of canonicity condition

$$\begin{aligned}
\{a_k, a_{k'}\} &= \int \left\{ \frac{\delta a_k}{\delta b_{k''}}\frac{\delta a_{k'}}{\delta b^*_{k''}} - \frac{\delta a_k}{\delta b^*_{k''}}\frac{\delta a_{k'}}{\delta b_{k''}} \right\} dk'' = 0 \\
\{a_k, a^*_{k'}\} &= \int \left\{ \frac{\delta a_k}{\delta b_{k''}}\frac{\delta a^*_{k'}}{\delta b^*_{k''}} - \frac{\delta a_k}{\delta b^*_{k''}}\frac{\delta a^*_{k'}}{\delta b_{k''}} \right\} dk'' = \delta(k-k')
\end{aligned} \tag{3.27}$$

Calculation of $a_k^{(3)}$ by this method is just impossibly complicated task. The canonical transformation can be found using more sophisticated methods. The first one was offered in the article [7] in 1998. Let us consider that a_k is a solution of Hamiltonian system

$$\frac{\partial a_k}{\partial \tau} + i\frac{\delta R}{\delta a^*_k} = 0 \tag{3.28}$$

where τ is "artificial time" and R is an efficient Hamiltonian

$$R = i\int \Gamma^{(1)}_{kk_2k_2}(a_k^* a_{k_1} a_{k_2} - a_k a_{k_1}^* a_{k_2}^*)\,\delta(k-k_1-k_2)\,dkdk_1dk_2 + \\ +\frac{i}{3}\int \Gamma^{(2)}_{kk_1k_2}(a_k^* a_{k_1}^* a_{k_2}^* - a_k a_{k_1} a_{k_2})\,\delta(k+k_1+k_2)\,dkdk_1dk_2 \qquad (3.29)$$

Eq. (3.28, 3.29) must be augmented with initial condition

$$a_k|_{\tau=0} = b_k. \qquad (3.30)$$

The needed canonical transformation is obtained if we put $\tau = 1$. Expanding the solution in Taylor series of τ and putting $\tau = 1$ at the end, we reproduce the result of Krasitski (3.24 - 3.26) in a much more economical way.

Now we demonstrate another, more traditional way for constructing of canonical transformation, which is based on finding of generating function. We present a_k in the form

$$a_k = \frac{1}{\sqrt{2}}(q_k + ip_k), \quad q_{-k} = q_k^*, \quad p_{-k} = p_k^*$$

Functions q_k, p_k obey equations

$$\frac{\partial q_k}{\partial t} = \frac{\delta H}{\delta p_k^*}, \quad \frac{\partial p_k}{\partial t} = -\frac{\delta H}{\delta q_k^*} \qquad (3.31)$$

where H is the same Hamiltonian expressed through q_k, p_k. Now

$$H_0 = \frac{1}{2}\int \omega_k(|q_k|^2 + |p_k|^2)dk \qquad (3.32)$$

$$H_1 = \frac{1}{2}\int L_{kk_1k_2}\, q_k\, p_{k_1}\, p_{k_2}\, \delta(k+k_1+k_2)dkdk_1dk_2 \qquad (3.33)$$

$$L_{kk-1k_2} = \frac{g^{1/4}\, A_k^{1/4}}{A_{k-1}^{1/4}\, A_{k_2}^{1/2}}\, L^{(1)}_{k_1k_2} \qquad (3.34)$$

We will perform transformation to new variables R_k, ξ_k using the following generation function (see also [10]):

$$S = \int R_k\, q_k\, dk + \frac{1}{2}\int A_{kk_1k_2}\, q_k\, q_{k_1}\, R_{k_2}\, \delta(k+k_1+k_2)\, dkdk_1dk_2 + \\ +\frac{1}{3}\int B_{kk_1k_2}\, R_k\, R_{k_1} R_{k_2}\, \delta(k+k_1+k_2)dkdk_1dk_2 \qquad (3.35)$$

The "old momentum" p_k and "new coordinates" ξ_k are expressed as follow

$$p_k = \frac{\delta S}{\delta q_{-k}} = R_k + \int A_{-k,k_1,k_2}\, q_{k_1}\, R_{k_2}\, \delta(k-k_1-k_2)dk_1dk_2 \qquad (3.36)$$

$$\xi_k = \frac{\delta S}{\delta R_{-k}} = q_k + \frac{1}{2}\int A_{k_1,k_2,-k}\, q_{k_1}\, q_{k_2}\, \delta(k-k_1-k_2^*)\, dk_1dk_2 + \\ + \int B_{-k,k_1,k_2}\, R_{k_1}\, R_{k_2}\, \delta(k-k_1-k-2)\, dk_1dk_2 \qquad (3.37)$$

Apparently $B_{kk_1k_2}$ is symmetric with respect to all permutations and $A_{kk_1k_2} = A_{kk_2k_1}$. To find A, B we notice that in the first approximation

$$\begin{aligned} q_k &= \xi_k - \frac{1}{2}\int A_{k_1,k_2,-k}\,\xi_{k_1}\,\xi_{k_2}\,\delta(k-k_1-k_2)\,dk_1dk_2 - \\ &\quad - \int B_{-k,k_1,k_2}\,R_{k_1}\,R_{k_2}\,\delta(k-k_1-k_2)\,dk_1dk_2 \end{aligned} \tag{3.38}$$

and in (3.36) we can replace $q_k \to \xi_k$. Now we plug q_k, p_k to (3.32). In (3.33) we can just replace $q_k \to \xi_k$ and $p_k \to R_k$. From the condition of eliminating cubic terms that are proportional to $\xi_k\xi_{k_1}\xi_{k_2}$ and $\xi_k p_{k_1} p_{k_2}$, and the symmetry conditions we find after some calculations the following nice and elegant expressions for A, B:

$$A_{kk_1k_2} = -\frac{1}{4}\left(\frac{L_0+L_1+L_2}{\omega_0+\omega_1+\omega_2} + \frac{L_0+L_1-L_2}{\omega_0+\omega_1-\omega_2}\right) + \frac{1}{4}\left(\frac{L_0-L_1-L_2}{\omega_0-\omega_1-\omega_2} + \frac{L_1-L_0-L_2}{\omega_1-\omega_0-\omega_2}\right) \tag{3.39}$$

$$B_{kk_1k_2} = -\frac{1}{4}\left(\frac{L_0+L_1+L_2}{\omega_0+\omega_1+\omega_2} + \frac{L_0-L_1-L_2}{\omega_0-\omega_1-\omega_2}\right) - \frac{1}{4}\left(\frac{L_1-L_0-L_2}{\omega_1-\omega_0-\omega_2} + \frac{L_2-L_0-L_1}{\omega_2-\omega_0-\omega_1}\right) \tag{3.40}$$

Here

$$\begin{aligned} &L_0 = L_{kk_1k_2}, \quad L_1 = L_{k_1kk_2}, \quad L_2 = L_{k_2kk_1} \\ &\omega_0 = \omega_k, \quad \omega_1 = \omega_{k_1}, \quad \omega_2 = \omega_{k_2} \end{aligned} \tag{3.41}$$

To reproduce the results of Krasitski one has to expand old variables q_k, p_k in powers of new variables ξ_k, R_k, then b_k as follow

$$b_k = \frac{1}{\sqrt{2}}\left(\left(\frac{g}{A_k}\right)^{1/4}\xi_k - i\left(\frac{A_k}{g}\right)^{1/4}R_k\right) \tag{3.42}$$

New normal variables b_k satisfy Zakharov's equation [6]

$$\frac{\partial b_k}{\partial t} + i\omega_k\,b_k + \frac{i}{2}\int T_{kk_1k_2k_3}\,b^*_{k_1}\,b_{k_2}\,b_{k_3}\,\delta_{k+k_1-k_2-k_3}\,dk_1dk_2dk_3 = 0 \tag{3.43}$$

Here $T_{kk_1k_2k_3}$ is the same as in (2.2). An explicit expression for $T_{kk_1k_2k_3}$ is too complicated to be presented here. Notice that now we can calculate $n_k = |a_k|^2$ by use of expansion (3.23). We will assume that triple correlations of new variables are zero

$$< b_k b_{k_1} b_{k_2} >= 0, \quad < b^*_k b_{k_1} b_{k_2} >= 0 \tag{3.44}$$

We use also the Gaussian closure for quartic variables

$$< b^*_k b^*_{k_1} b_{k_2} b_{k_3} >= N_k\,N_{k_1}(\delta_{k-k_2}\,\delta_{k_1-k_3} + \delta_{k-k_3}\,\delta_{k_1-k_2}) \tag{3.45}$$

Here N_k is the "refined" action. After some calculations we find that n_k and N_k are connected by the following relation (it can be found in [8]):

$$\begin{aligned} n_k &= N_k + \frac{1}{2}\int \frac{|V^{(1,2)}(\vec{k},\vec{k}_1,\vec{k}_2)|^2}{(\omega_k-\omega_{k_1}-\omega_{k_2})^2}\left(N_{k_1}N_{k_2}-N_kN_{k_1}-N_kN_{k_2}\right)\delta(\vec{k}-\vec{k}_1-\vec{k}_2)\,dk_1dk_2 + \\ &+\frac{1}{2}\int \frac{|V^{(1,2)}(\vec{k},\vec{k}_1,\vec{k}_2)|^2}{(\omega_{k_1}-\omega_k-\omega_{k_2})^2}\left(N_{k_1}N_{k_2}+N_kN_{k_1}-N_kN_{k_2}\right)\delta(\vec{k}_1-\vec{k}-\vec{k}_2)\,dk_1dk_2 + \\ &+\frac{1}{2}\int \frac{|V^{(1,2)}(\vec{k}_2,\vec{k},\vec{k}_1)|^2}{(\omega_{k_2}-\omega_k-\omega_{k_1})^2}\left(N_{k_1}N_{k_2}+N_kN_{k_2}-N_kN_{k_1}\right)\delta(\vec{k}_2-\vec{k}-\vec{k}_1)\,dk_1dk_2 + \\ &+\frac{1}{2}\int \frac{|V^{(0,3)}(\vec{k},\vec{k}_1,\vec{k}_2)|^2}{(\omega_k+\omega_{k_1}+\omega_{k_2})^2}\left(N_{k_1}N_{k_2}+N_kN_{k_1}+N_kN_{k_2}\right)\delta(\vec{k}+\vec{k}_1+\vec{k}_2)\,dk_1dk_2 \end{aligned} \tag{3.46}$$

The difference between n_k and N_k,

$$\Delta_k = \frac{n_k - N_k}{N_k},$$

is essential on shallow water. However, even on deep water Δ_k is a fast growing function on k.

The relation between space-time spectra of "total" $n_{k\omega}$ and "purified" $N_{k\omega}$ versions of wave action is not known so far. This is a subject for future research. However, $N_{k\omega}$ can be presented in the form

$$N_{k\omega} = \frac{1}{\pi}\frac{\Gamma_k\, N_k}{(\omega-\tilde{\omega}_k)^2+\Gamma_k^2} \tag{3.47}$$

and we can put approximately

$$Q_{k\omega} = \frac{1}{2}\,\omega_k(N_{k\omega}+N_{-k,-\omega}) = \frac{1}{2\pi}\left\{\frac{\Gamma_k\, N_k}{(\omega-\tilde{\omega}_k)^2+\Gamma_k^2} + \frac{\Gamma_{-k}\, N_{-k}}{(\omega-\tilde{\omega}_k)^2+\Gamma_k^2}\right\} \tag{3.48}$$

After integration by ω and assuming that $\arctan \Gamma_k/\omega_k \sim \Gamma_k/\omega_k$, one gets the following relation

$$N_k = \int_0^\infty N(k,\omega)d\omega + \frac{1}{\pi}\left(\frac{N_k\,\Gamma_k}{\omega_k} - \frac{N_{-k}\,\Gamma_{-k}}{\omega_{-k}}\right) \tag{3.49}$$

From (3.48) we see that identity

$$N_k = \int_0^\infty N(k,\omega)\,d\omega \tag{3.50}$$

is valid up to the relative accuracy Γ_k/ω_k. The value of this accuracy will be discussed in Chapter 6. Near the spectral peak it is of order $4\pi\mu^4$. Identity (2.18) is satisfied with much less accuracy. Even near the spectral peak the accuracy is of order μ^2 and it becomes worse at $k \gg k_p$. An explicit expression for $Q(k,\omega)$ through N_k will be the subject of a separate article.

4 Stationary solutions of kinetic equation: Isotropic case

In this chapter we address the following question: How to solve the stationery kinetic equation

$$S_{nl} \equiv 0\,? \tag{4.1}$$

Formally speaking, this equation has thermodynamically equilibrium solutions

$$N_k = \frac{T}{\omega_k + \mu}\,, \tag{4.2}$$

where temperature T and μ are constants. It might sound like paradox, but in fact spectrum (4.2) in not a real solution of equation (4.1). Since this moment we discuss only the case of deep water and consider $\omega = \sqrt{gk}$. Also we denote that $k = |\vec{k}|$.

To justify this statement we notice that in two particular cases, $\mu = 0$ and $T = c\mu$, $\mu \to \infty$, solution (4.2) takes form

$$\begin{aligned} N &= \frac{T}{\omega_k} = \frac{T}{\sqrt{g}}\, k^{-1/2} \\ N &= c \end{aligned} \tag{4.3}$$

Both these solutions are isotropic powerlike functions

$$N_k = k^{-x} \tag{4.4}$$

with particular values $x = 1/2, 0$. Let us study the general powerlike solution of (4.1). By plugging (4.4) into (4.1) we find that each particular term in S_{nl} is diverging, but in different terms the divergence can be cancelled, thus there is a "window of opportunity" for the exponent x. As a result,

$$S_{nl} = g^{3/2}\, k^{-3x+19/2}\, F(x). \tag{4.5}$$

Here $F(x)$ is a dimensionless function, defined inside interval $x_1 < x < x_2$. The edges of the window, x_1 and x_2, are the subject for determination. Outside the "window of opportunity", at $x < x_1$ and $x > x_2$, $F(x) = \infty$. Thus all admitted values of x must be posed between x_1 and x_2.

Let the quadruplet of waves be formed of wave vectors satisfying resonant conditions

$$\begin{aligned} \vec{k}_1 + \vec{k}_2 &= \vec{k}_3 + \vec{k}_4 \\ \omega_{k_1} + \omega_{k_2} &= \omega_{k_3} + \omega_{k_4} \end{aligned} \tag{4.6}$$

Suppose that $|k_1| \ll |k|$. The three-wave resonant condition,

$$\vec{k} = \vec{k}_2 + \vec{k}_3, \qquad \omega_k = \omega_{k_2} + \omega_{k_3}, \tag{4.7}$$

can not be satisfied, thus one of vectors $\vec{k}_2, \vec{k}_3$ must be small. If $|k_3| \ll |k_2|$, then

$$\begin{aligned} \vec{k}_2 &= \vec{k} + \vec{k}_1 - \vec{k}_3, \\ \omega(k_2) &= \sqrt{gk}\left(1 + \frac{1}{2}\frac{(k, \vec{k}_1 - \vec{k}_3)}{k^2} + \cdots\right) \end{aligned} \tag{4.8}$$

In the first approximation by small parameter $|k_1|/|k|$ one can put $\omega(k_2) = \omega(k)$, $\omega(k_1) = \omega(k_3)$ and $|k_3| \simeq |k_1|$. In other words, vectors $\vec{k}_1, \vec{k}_3$ are small and have approximately the same length k_1. If vector k is directed along axis x, the coupling coefficient $T_{kk_1k_2k_3}$ depends on four parameters $k, k_1, \theta_1, \theta_3$. Here θ_1, θ_3 are angles between $\vec{k}_1, \vec{k}_3$ and $\vec{k}$. Remembering that $k_1 \ll k$, we calculate the coupling coefficient in this asymptotic domain. A tedious calculation presented in [11] leads to the following compact result:

$$\begin{aligned} T_{kk_1k_2k_3} &\simeq \frac{1}{2} k\, k_1^2\, T_{\theta_1,\theta_3}, \\ T_{\theta_1,\theta_2} &= 2(\cos\theta_1 + \cos\theta_3) - \sin(\theta_1 - \theta_3)(\sin\theta_1 - \sin\theta_3). \end{aligned} \tag{4.9}$$

On the diagonal $k_3 = k_1$, $\theta_3 = \theta_1$ we get a very simple expression published in 2003 [29]:

$$T_{kk_1} \simeq 2k_1^2 k\, \cos\theta_1. \tag{4.10}$$

Suppose that spectrum is separated to the low-frequency component $N_0(k)$ and the high-frequency component $N_1(k)$. We assume that $N_1 \ll N_0$ and take into account the interaction between N_0 and N_1 only. One can see that N_1 satisfies the linear diffusion equation

$$\frac{\partial}{\partial t} N_1 = \frac{\partial}{\partial k_i} D_{ij}\, k^2\, \frac{\partial}{\partial k_j} N_1, \tag{4.11}$$

where D_{ij} is the tensor of diffusion coefficients,

$$D_{ij} = 2\pi g^{3/2} \int_0^\infty dq\, q^{17/2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_3 |T(\theta_1, \theta_3)|^2\, p_i p_j N(\theta, q) N(\theta_3, q) \tag{4.12}$$

$$p_1 = \cos\theta_1 - \cos\theta_3, \quad p_2 = \sin\theta_1 - \sin\theta_3$$

If spectrum is isotropic and does not depend on angle θ, we get the further simplification:

$$D_{ij} = D\,\delta_{ij}, \qquad D = \frac{5}{8}\pi^3\, g^{3/2} \int_0^\infty q^{17/2}\, N^2(q) dq. \tag{4.13}$$

The diffusion coefficient D diverges at $k \to 0$, if $x > 19/4$. Thus $x_2 = 19/4$.

Let us find behavior of function $F(x)$ near $x = x_2$. In the isotopic case equation (3.9) reads

$$\frac{\partial N_1}{\partial t} = \frac{D}{k}\frac{\partial}{\partial k} k^3 \frac{\partial}{\partial k} N_1. \tag{4.14}$$

If $k \to 19/4$, we get the following estimate:

$$F(x) = \frac{19}{4} \cdot \frac{11}{4} \cdot \frac{5\pi^3}{16} \, \frac{1}{19/4 - x} \simeq \frac{126.4}{19/4 - x} \tag{4.15}$$

To find x_1, the lower end of the window, we should study the influence of short waves to the long ones. Let us suppose that $|k_1|, |k_2| \gg k$. In the first approximation $|k_3| = |k|$, and the resonant interaction S_{nl} can be separated into two groups of terms: $S_{nl} = S_{nl}^{(1)} + S_{nl}^{(2)}$. For $S_{nl}^{(1)}$ the integrand includes product $N_{k_1} \, N_{k_2}$. If we put $k_1 = k_2$, we get the following expression for the low-frequency tail of spectrum:

$$S_{nl}^{(1)} = 2\pi g^2 \int |T_{kk_1,k_1,k_3}|^2 \, \delta(\omega - \omega_{k_3}) \, (N_{k_3} - N_k) \, N_{k_1}^2 \, dk_1. \tag{4.16}$$

Notice, if $|k_1| \gg |k|$, then $|T_{kk_1,k_1,k_3}|^2 \simeq k_1^2$, and integrand in (4.16) is proportional to $k_1^2 \, N_{k_1}^2$. If $x < 2$, the integral diverges.

The group of terms linear with respect to the high-frequency tail of spectrum is more complicated:

$$\begin{aligned} S_{nl}^{(2)} &= 2\pi g^2 \, N_k \int |T_{kk_1k_2k_3}|^2 \, N_{k_3} (N_{k_1} - N_{k_2}) \times \\ &\times \; \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \, \delta(k + k_1 - k_2 - k_3) \, dk_1 \, dk_2 \, dk_3. \end{aligned} \tag{4.17}$$

We can perform expansion

$$N_{k_1} - N_{k_3} = p_i \, \frac{\partial N}{\partial k_{1i}}, \quad p_i = (k - k_3)_i. \tag{4.18}$$

In the general anisotropic case the integrand is proportional to $k_1^2 (p \, \nabla \, N_{k_1})$ and the divergence occurs if $x = x_1 = 3$. However, in the isotropic case this term, the most divergent one, is cancelled after integration by angles. In this case we should study quadratic terms in expansion of the integrand in powers of parameter $(P, k_1)/k_1^2$. The most aggressive term appears from the expansion of δ-function on frequencies $\delta(\omega_{k_1} - \omega_{k_1+p} + \omega_k - \omega_{k_3})$. Performing integration by angles we end up with equation

$$\begin{aligned} \frac{\partial N_k}{\partial t} &= q \, k^7 \, N_k \, \frac{\partial N}{\partial k}, \\ q &= \frac{25}{16} \pi^3 \, g^{3/2} \, E = \frac{25}{8} \pi^3 \, g^{3/2} \int_0^\infty k^{3/2} \, N_k \, dk. \end{aligned} \tag{4.19}$$

Here E is the total energy. Thus in the isotropic case $x_1 = 5/2$ and we get for function $F(x)$ the following estimate:

$$F = \frac{5}{2} \, \frac{25}{8} \, \pi^3 \, \frac{1}{5/2 - x} = \frac{241.86}{5/2 - x}. \tag{4.20}$$

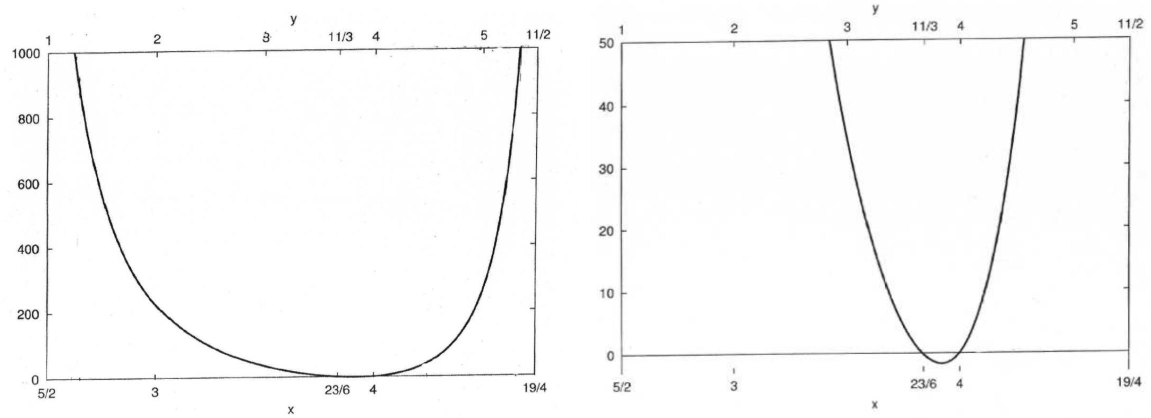

Figure 1: (a) Plot of function $F(x)$. (b) Plot of function $F(x)$: zoom in the vertical direction

On Figure 1a is presented the plot of function $F(x)$ for isotropic case that we calculated numerically. One can see that in the interval $x_1 < x < x_2$ function $F(x)$ has exactly two zeros at

$$x = y_1 = 4, \quad x = y_2 = \frac{23}{6}. \tag{4.21}$$

To prove this result, let us consider that spectra are isotropic and present conservation laws of energy and wave action in the differential form:

$$\frac{\partial I_k}{\partial t} = 2\pi k\omega_k \frac{\partial N_k}{\partial t} = -\frac{\partial P}{\partial k}, \tag{4.22}$$

$$P = 2\pi \int_0^k k\omega_k \, S_{nl} \, dk, \tag{4.23}$$

$$2\pi k \frac{\partial N_k}{\partial t} = \frac{\partial Q}{\partial k}, \tag{4.24}$$

$$Q = 2\pi \int_0^k k \, S_{nl} \, dk. \tag{4.25}$$

Here P is the flux of energy directed to high wave numbers, while Q is the flux of wave action directed to small wave numbers. Equations

$$P = P_0 = const, \quad Q = Q_0 = const \tag{4.26}$$

apparently are solutions of stationary equation $S_{nl} = 0$. We will look for the solution in the powerlike form $N = \lambda\, k^{-x}$; then equations (4.23), (4.25) read

$$P_0 = 2\pi g^2 \lambda^3 \frac{F(x)}{3(x-4)} k^{-3(x-4)} \tag{4.27}$$

$$Q_0 = -2\pi g^{3/2} \lambda^3 \frac{F(x)}{3(x-26/3)} k^{-3(x-26/3)} \tag{4.28}$$

One can see that P_0 and Q_0 are finite only if $F(4) = 0$ and $F(26/3) = 0$, moreover, if $F'(4) > 0$ and $F'(26/3) < 0$. We conclude that equation $S_{nl} = 0$ has the following solutions:

$$N_k^{(1)} = c_p \left(\frac{P_0}{g^2}\right)^{1/3} \frac{1}{k^4}, \tag{4.29}$$

$$N_k^{(2)} = c_q \left(\frac{Q_0}{g^{3/2}}\right)^{1/3} \frac{1}{k^{23/6}}. \tag{4.30}$$

Here c_p, c_q are dimensionless Kolmogorov constants

$$c_p = \left(\frac{3}{2\pi\, F'(4)}\right)^{1/3}, \quad c_q = \left(\frac{3}{2\pi\, |F'(23/6)|}\right)^{1/3}.$$

On Figure 1b is presented the zoom of function $F(x)$ in vertical coordinate. The numerics gives $F'(4) = 45.2$ and $F'(23/6) = -40.4$. In the area of zeros $F(x)$ can be approximated by parabola,

$$F(x) \simeq 256.8(x - 23/6)(x - 4). \tag{4.31}$$

Let us notice that

$$F(9/2) = 85.6 \tag{4.32}$$

thus we get

$$c_p = 0.219, \quad c_q = 0.227, \tag{4.33}$$

and see that the both Kolmogorov constants are numerically small.

In the isotropic case, the energy spectrum $F(\omega)$ can be expressed through N_k,

$$F(\omega) d\omega = 2\pi\omega_k\, N_k\, k\, dk, \tag{4.34}$$

and the energy spectrum corresponding to solution (4.29) has the following form, called Zakharov-Filonenko spectrum:

$$F^{(1)}(\omega) = 4\pi c_p \left(\frac{P}{g^2}\right)^{1/3} \frac{g^2}{\omega^4}. \tag{4.35}$$

This spectrum was found as a solution of equation $S_{nl} = 0$ [3]. For the spatial spectrum

$$I_k\, dk = 2\pi\omega_k\, N(k)\, k\, dk, \tag{4.36}$$

solution (4.30) transforms to

$$I_k^{(1)} = 2\pi\, c_p \left(\frac{P}{g^2}\right)^{1/3} \frac{g^{1/2}}{k^{5/2}} \simeq k^{-2.5}. \tag{4.37}$$

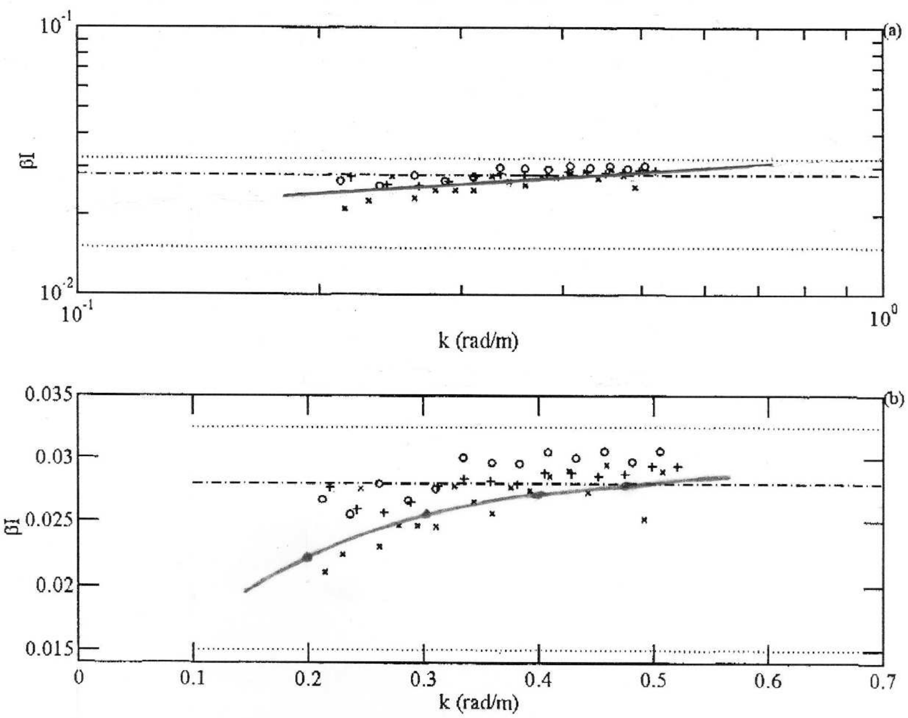

Figure 2: Dimensionless wavenumber spectral coefficient β_i plotted in logarithmic scales (a) and linear scales (b), taken from [20]. Here crosses represent omnidirectional (averaged by angles) spectrum and dots correspond to $\xi(k) = 2\beta_I\, u_*\, g^{-0.5}\, k^{-2.5}$. The solid line on (a) and solid curve on (b) correspond to $\xi(k) \simeq k^{-7/3}$.

Spectra (4.29), (4.35), (4.37) are realized if we have a source of energy that is concentrated at small wave number and generates the amount of energy P in a unit of time. For the spectrum (4.30), first reported by Zakharov in 1966 [3],

$$I_k^{(2)} = 2\pi\, c_q\, Q^{1/3}\, k^{-7/3} \simeq 2\pi\, c_q\, Q^{1/3}\, k^{2.33}, \tag{4.38}$$

$$F^{(2)}(\omega) = 4\pi\, c_q\, Q^{1/3}\, \frac{g^{4/3}}{\omega^{11/3}}. \tag{4.39}$$

Spectra (4.30) and (4.38) can be realized in the case of source of wave action in the high wave numbers area.

The described spectra exhaust all powerlike isotropic solutions of the stationary kinetic equation $S_{nl} = 0$. It is important to stress that thermodynamical solutions $N = const$ and $N = c/k^{1/2}$ are not the solutions of this equation, because their exponents $x = 0$ and $x = 1/2$ are far below the lower end of the "window of possibility" $x_1 = 5/2$. This fact means that thermodynamics has nothing in common with the theory of wind-driven sea.

Solutions (4.29) and (4.30) are not the unique stationary solutions of $S_{nl} = 0$. The general isotropic solution describes the situation when both the energy source at small wave numbers and the wave action source exist simultaneously and have the following form:

$$N_k^{(3)} = c_p \left(\frac{P}{g^2}\right)^{1/3} \frac{1}{k^4} L \left(\frac{g^{1/2} Q\, k^{1/2}}{P}\right). \tag{4.40}$$

Here L is an unknown function of one variable,

$$L \to 1 \quad \text{at} \quad k \to 0, \qquad L(\xi) \to \frac{c_q}{c_p}\, \xi^{1/3} \quad \text{at} \quad k \to \infty. \tag{4.41}$$

Let us notice that if there is no flux of wave action from infinity, we must put $Q = 0$. Under this constrain, the general isotropic solution is the Zakharov-Filonenko spectrum (4.29), parametrized by a single arbitrary constant P, which is a flux of energy to $k \to \infty$.

Frequency spectra with tails in the form $F(\omega) \simeq \omega^{-4}$ were observed in numerous field experiments [11-16] and were obtained in numerical experiments as well [17-19]. Spatial spectra with asymptotics $I_k \simeq k^{5/2}$ were observed also in many experiments [20-22]. A more careful study of experimental results show that in the majority of cases the spectral area right behind the spectral peak can be better approximated by tail $\omega^{-11/3}$ in frequency spectrum and by tail $k^{-7/3}$ in spatial spectrum. It is seen especially clear in the experiments by Huang and collaborators]20]. Figure 2 taken from this article demonstrates coexistence of both types of KZ spectra.

5 Stationary solutions of kinetic equation: Anisotropic case

To study anisotropic solutions of equation (4.1) we introduce polar coordinates on k-plane and put $k^2 = \omega/g$. Thereafter we will use notation

$$\begin{aligned} N(\omega,\phi)\, d\omega d\phi &= N(\vec{k})\, d\vec{k}, \\ N(\omega,\phi) &= \frac{2\omega^3}{g^2}\, N(\vec{k}). \end{aligned} \tag{5.1}$$

In the spatially homogenous case $N(\omega,\phi)$ satisfies equation

$$\frac{\delta N(\omega,\phi)}{\partial t} = S_{nl}(\omega,\phi). \tag{5.2}$$

In new variables:

$$\begin{aligned} S_{nl}(\omega,\phi) \;=\; & 2\pi g^2 \int |T_{\omega,\omega_1,\omega_2,\omega_3}|^2\, \delta(\omega+\omega_1-\omega_2-\omega_3) \times \\ & \times \delta(\omega^2\cos\phi + \omega_1^2\cos\phi_1 - \omega_2^2\cos\phi_2 - \omega_3^2\cos\phi_3) \times \\ & \times \delta(\omega^2\sin\phi + \omega_1^2\sin\phi_1 - \omega_2^2\sin\phi_2 - \omega_3^2\sin\phi_2) \times \\ & \times \Big\{ \omega^3\, N(\omega_1,\phi_1)\, N(\omega_2,\phi_2)\, N(\omega_3,\phi_3) + \omega_1^3\, N(\omega,\phi)\, N(\omega_2,\phi_2)\, N(\omega_3,\phi_3) - \\ & - \omega_2^2\, N(\omega,\phi)\, N(\omega_1,\phi_1)\, N(\omega_3,\phi_3) - \omega_3^2\, N(\omega,\phi)\, N(\omega_1,\phi_1)\, N(\omega_2,\phi_2) \Big\} \\ & \qquad d\omega_1\, d\omega_2\, d\omega_3\, d\phi_1\, d\phi_2\, d\phi_3. \end{aligned} \tag{5.3}$$

Exactly this form of S_{nl} is used for numerical simulation of Hasselmann equation. Suppose that $N(\omega,\phi)=\omega^{-z}$ is isotropic spectrum. Then

$$S_{nl}=\frac{\omega^{-3z+13}}{4g^4}\,F\left(\frac{z+3}{2}\right)=\frac{G(z)}{g^4}\,\omega^{-3z+13}, \tag{5.4}$$

where $F(x)$ is defined by (4.5). Now the "window of opportunity" is: $2<z<13/2$. Zeros of $G(z)$ are posed at $z_1=5$ and $z_2=14/3$ and near these zeros $G(z)$ can be presented as parabola,

$$G(z)\simeq\,16.05(z-5)(z-14/3). \tag{5.5}$$

To make the motion constants more conspicuous, we introduce the elliptic differential operator

$$L\,f(\omega,\phi)=\left(\frac{\partial^2}{\partial\omega^2}+\frac{2}{\omega^2}\,\frac{\partial^2}{\partial\phi^2}\right)\,f(\omega,\phi) \tag{5.6}$$

with following parameters: $0<\omega<\infty,\;\;0<\phi<2\pi$. Equation

$$L\,G=\delta(\omega-\omega')\,\delta(\phi-\phi') \tag{5.7}$$

with boundary conditions

$$G|_{\omega\to 0}=0,\qquad G_{\omega\to\infty}<\infty,\qquad G(2\pi)=G(0),$$

can be resolved as

$$G(\omega,\omega',\phi-\phi')=\frac{1}{4\pi}\sqrt{\omega\omega'}\sum_{n=-\infty}^{\infty}e^{in(\phi-\phi')}\times$$
$$\times\left[\left(\frac{\omega}{\omega'}\right)^{\Delta_n}\,\Theta(\omega'-\omega)+\left(\frac{\omega'}{\omega}\right)^{\Delta_n}\,\Theta(\omega-\omega')\right], \tag{5.8}$$

where $\Delta_n=1/2\sqrt{1+8n^2}$. Now we present S_{nl} in the form:

$$A(\omega,\phi)=\int_0^\infty d\omega'\int_0^{2\pi}d\phi'\,G(\omega,\omega',\phi-\phi')\,S_{nl}(\omega',\phi'). \tag{5.9}$$

Notice that $A(\omega,\phi)$ is a regular integral operator and suppose that $N(\omega,\phi)=\omega^{-z}$. Then

$$A[\omega^{-z}]=\frac{\omega^{-3z+15}}{g^4}\,H(z),$$
$$H(z)=\frac{G(z)}{9(z-5)(z-14/3)}. \tag{5.10}$$

Function $H(z)$ is positive and has no zeros. If $G(z)$ is presented by parabola (5.5), $H(z)$ is just a constant:

$$H(z)=H_0=16.05/9=1.83. \tag{5.11}$$

This fact leads to a bold idea. If we assume that

$$A = \frac{H_0}{g^4}\,\omega^{15}\,N^3, \tag{5.12}$$

the nonlinear term S_{nl} turns to the elliptic operator:

$$S_{nl} = \frac{H_0}{g^4}\left(\frac{\partial^2}{\partial\omega^2} + \frac{2}{\omega^2}\frac{\partial^2}{\partial\phi^2}\right)\omega^{15}\,N^3. \tag{5.13}$$

This is a so-called "diffusion approximation", introduced in article [23]. Being very simple, it grasps the basic features of wind-driven sea theory. We will refer mostly to this model, having in mind that the real case (5.9) does not differ much from it, at least qualitatively.

Let us integrate equation (5.2) by angles. We get:

$$\frac{\partial N(\omega,t)}{\partial t} = \frac{\partial Q}{\partial\omega}. \tag{5.14}$$

Here $N(\omega,t) = \int_0^{2\pi} N(\omega,\phi)\,d\phi$. Then

$$B(\omega,t) = \frac{g}{2\omega}\int_0^{2\pi}\cos\phi\,N(\omega,\phi)\,d\phi, \tag{5.15}$$

and the flux of wave action is:

$$Q = \frac{\partial K}{\partial\omega}, \qquad K = \int_0^{2\pi} A(\omega,\phi)\,d\phi. \tag{5.16}$$

After multiplication of equation (5.14) by ω one obtains equation

$$\frac{\partial F(\omega,t)}{\partial t} + \frac{\partial P}{\partial\omega} = 0, \tag{5.17}$$

where $P = K - \omega\,\partial K/\partial\omega$ is the flux of energy.

Let us introduce now the following definitions: the integrated by angle spectral density of momentum

$$M_x(\omega,t) = \frac{\omega^2}{g}\int_0^{2\pi}\cos\phi\,B(\omega,\phi)\,d\phi, \tag{5.18}$$

the quantity

$$C_x(\omega,t) = \frac{\omega}{2g}\int_0^{2\pi}\cos^2\phi\,N(\omega,\phi)\,d\phi, \tag{5.19}$$

and the flux of momentum

$$R_x = \int_0^{2\pi}\cos\phi(\omega\,A - \frac{\omega^2}{2}\frac{\partial A}{\partial\omega})\,d\phi. \tag{5.20}$$

All these quantities are connected by equation

$$\frac{\partial M_x}{\partial t} + \frac{\partial R_x}{\partial\omega} = 0. \tag{5.21}$$

Equations (5.14), (5.17) and (5.21) are averaged by angle balance equations for the basic conservative quantities.

Now we can return to the question formulated above. How many solutions has the stationary kinetic equation (1.5), (4.1)? Notice that we simplified it to the linear equation

$$\left(\frac{\partial^2}{\partial\omega^2} + \frac{2}{\omega^2}\frac{\partial^2}{\partial\phi^2}\right) A = 0. \tag{5.22}$$

In particulary, kinetic equation has anisotropic KZ solution

$$A = \frac{1}{2\pi}\left\{P + \omega\, Q + \frac{R_x}{\omega}\cos\phi\right\}, \tag{5.23}$$

where P and R_x are fluxes of energy and momentum at $\omega \to \infty$ and Q is the flux of wave action directed to small wave numbers. In a general case, (5.23) is a nonlinear integral equation, however in the diffusion approximation the KZ solution can be found in the explicit form:

$$N(\omega,\phi) = \frac{1}{(2\pi\, H_0)^{1/3}}\,\frac{g^{4/3}}{\omega^5}\left(P + \omega\, Q + \frac{R_x}{\omega}\cos\phi\right)^{1/3}. \tag{5.24}$$

By comparison with (4.35), (4.38) we easily find that in this case

$$c_p = c_q = \frac{1}{2(2\pi\, H_0)^{1/3}} = 0.223, \qquad H_0 = 1.83.$$

This is exactly the arithmetic mean between the values of Kolmogorov constants given by (3.31).

By multiplication of (5.24) to $2\pi\omega$ we get the general KZ spectrum in the diffusion approximation:

$$F(\omega) = 2.78\,\frac{g^{4/3}}{\omega^4}\left(P + \omega\, Q + \frac{R_x}{\omega}\cos\phi\right)^{1/3}. \tag{5.25}$$

We must be sure that in the isotropic case $R_x = 0$, expression

$$F(\omega) = 2.78\,\frac{g^{4/3}}{\omega^4}\,(P + \omega\, Q)^{1/3} \tag{5.26}$$

approximates the generic KZ spectrum with accuracy up to few percent.

If somehow we know the value of $A(\omega,\phi)$ on the circle $\omega = \omega_0$, we can solve the external and internal Dirichlet boundary problem for equation (5.22) with boundary condition $A(\omega,\phi) < \infty$ at $\omega \to \infty$. Suppose that

$$A(\omega,\phi) = A_0(\phi) = A_0 + \frac{A_1}{\omega}\cos\phi + \sum_{n=2}^{\infty} A_n\left(\frac{\omega_0}{\omega}\right)^{-1/2+\sqrt{1/4+4n^2}}\cos n\phi. \tag{5.27}$$

First two terms in (5.27) present the KZ spectrum with $Q = 0$, $P = 2\pi A_n$, $R_x = 2\pi\omega_0\, A_1$. The next terms describe the fast stabilization of any arbitrary solution to the KZ spectrum at $\omega/\omega_0 \to \infty$. The first additional term in (5.27) decays as $(\omega_0/\omega)^{3.53}\cos 2\phi$.

This stabilization to KZ spectrum is actually the "angular spreading" of wind-driven wave spectra that is usually observed in field experiments (see, for instance [12]). If $Q = 0$, the general KZ solution (5.25) at $\omega \to 0$ is the following spectrum:

$$F(\omega) \to \frac{2.78}{\omega^4} g^{4/3} p^{1/3} \left(1 + \frac{1}{3} \frac{R_x}{P\,\omega} \cos\phi + \cdots\right). \tag{5.28}$$

Similar results were predicted by Kontorovich and Kats [30] and Balk [31].

From (5.27) one can see that $A(\omega, \phi)$ is parametrized by function of one variable, $A_0(\phi)$. In presence of flux of action Q from infinity one should add to (5.27) an additional term Q_ω. Thus in a general case, a freedom for determination of A consists of the function that has one variable and one constant. We silently assume that the mapping $N \to A$ is uniquely inversible. This fact is not proven but it is very plausible.

6 Damping due to nonlinear interaction

How we must compare S_{nl} and S_{in}?

In this Chapter we show that S_{nl} is the leading term in the balance equation (1.11). In fact, the forcing terms S_{in} and S_{dis} are not known well enough, thus it is reasonable to accept the most simple models of both terms assuming that they are proportional to the action spectrum:

$$S_{in} = \gamma_{in}(k)\, N(k), \tag{6.1}$$
$$S_{dis} = -\gamma_{dis}(k)\, N(k). \tag{6.2}$$

Hence

$$\gamma(k) = \gamma_{in}(k) - \gamma_{dis}(k). \tag{6.3}$$

In reality $\gamma_{dis}(k)$ depends dramatically on the overall steepness μ. So far, let us notice that the balance kinetic equation (1.24) can be written in the form

$$S_{nl} + \gamma(k)\, N_k = 0, \tag{6.4}$$

and present the S_{nl} term as

$$S_{nl} = F_k - \Gamma_k\, N_k \tag{6.5}$$

The definition of Γ_k and F_k are given by (2.5), (2.6).

The solution of stationary equation (6.4) is the following:

$$N_k = \frac{F_k}{\Gamma_k - \gamma_k}. \tag{6.6}$$

The positive solution exists if $\Gamma_k > \gamma_k$. The term Γ_k can be treated as the nonlinear damping that appear due to four-wave interaction. This damping has a very powerful effect. A "naive" dimensional consideration gives

$$\Gamma_k \simeq \frac{4\pi g^2}{\omega_k} k^{10} N_k^2, \tag{6.7}$$

however, this estimate works only if $k \simeq k_p$; k_p being the wave number of the spectral maximum.

Let $k \gg k_p$. Now for Γ_k one gets

$$\Gamma_k = 2\pi g^2 \int |T_{kk_1,kk_3}|^2\, \delta(\omega_{k_1} - \omega_{k_3})\, N_{k_1} N_{k_3}\, dk_1 dk_2. \tag{6.8}$$

The main source of Γ_k is the interaction of long and short waves. To estimate integral (2.6) more accurately, we assume that the spectrum of long waves is narrow in angle, $N(k_1, \theta_1) = \tilde{N}(k_1)\, \delta(\theta_1)$. Long waves propagate along the axis x and $\vec{k}$ is the wave vector of short wave propagating in direction θ. For the coupling coefficient we must put $T_{kk_1,k_2,k_3} \simeq 2k_1^2 k \cos\theta$. Then

$$\Gamma_k = 8\pi g^{3/2}\, k^2\, \cos^2\theta \int_0^\infty k_1^{13/2}\, \tilde{N}^2(k_1)\, dk_1. \tag{6.9}$$

Even for the most mildly decaying KZ spectrum, $N_k \simeq k^{-23/6}$, the integrand behaves like $k_1^{-7/6}$ and the integral diverges. For more steep KZ spectra the divergence is stronger.

Let us estimate Γ_k for the case of "mature sea", when the spectrum can be taken in the form

$$N_k \simeq \frac{3}{2} \frac{E}{\sqrt{g}} \frac{k_p^{3/2}}{k^4}\, \theta(k - k_p). \tag{6.10}$$

Here E is the total energy. By plugging (6.10) to (6.9) one gets equation

$$\Gamma_\omega = 36\,\pi\omega \left(\frac{\omega}{\omega_p}\right)^3 \mu_p^4\, \cos^2\theta, \tag{6.11}$$

that includes a huge enhancing factor: $36\pi \simeq 113.04$. For the very modest value of steepness, $\mu_p \simeq 0.05$, we get

$$\Gamma_\omega \simeq 7.06 \cdot 10^{-4}\omega \left(\frac{\omega}{\omega_p}\right)^3 \cos^2\theta. \tag{6.12}$$

In the isotropic case, to find Γ_k for $\omega/\omega_p \gg 1$ we need to perform simple integration over angles that yields

$$\int_0^{2\pi}\int_0^{2\pi} T^2_{\theta_1,\theta_2}\, d\theta_1\, d\theta_2 = \frac{5}{2}(2\pi)^2,$$

thus instead of (6.11) we get:

$$\Gamma_k = 5\pi g^{3/2} k^2 \int_0^\infty k_1^{13/2} \tilde{N}(k_1)^2\, dk_1 \tag{6.13}$$

or

$$\Gamma_\omega = \frac{45\pi}{2} g^{3/2}\omega \left(\frac{\omega}{\omega_p}\right)^3 \mu_p^4. \tag{6.14}$$

Finally, assuming that

$$N_{k_p} \simeq \frac{3}{2}\frac{E}{\sqrt{g}k_p^{5/2}},$$

we get from (6.8) the following estimate for $\Gamma_p = \Gamma|_{k=kp}$:

$$\Gamma_p \simeq 9\pi\omega_p\mu_p^4. \tag{6.15}$$

Even in this case we have a pretty high enhancing factor: $9\pi \simeq 28.26$. In fact in all known models Γ_k surpasses $\tilde{\gamma}_k$ at least in order of magnitude even for these very smooth waves.

In the presence of peakedness

$$\Gamma_p \simeq \Lambda\,\omega_p\mu_p^4. \tag{6.16}$$

Here $\Lambda \simeq 4\pi\omega_p/\delta\omega$ is the enhancing factor due to peakedness. If $\Lambda\,\mu_p^2 \sim 1$, then Γ_p is associated with the maximal growth of modulational instability for monochromatic wave: $\Gamma_p \simeq \gamma_{mod} \sim \omega_p\mu_p^2$. If $\Lambda \sim 1/\mu_p^2$, the nonlinearity becomes so strong that the weak-turbulent statistical approach is not applicable. This is quite realistic situation. Suppose that $\mu_p \simeq 0.11$ and $\omega_p/\delta\omega \simeq 5$. Then $\Lambda\,\mu_p^2 \sim 0.76$ and the weak turbulent description is hardly correct. In the situation of strong nonlinearity the wind-driven sea generates freak waves (see [24, 25]). The very fact of their existence as a common phenomenon is an implicit proof of S_{nl} domination in the energy balance.

Notice that Γ_k diverges for KZ spectra. However, it does not hurt the spectra existence because in the full kinetic equation the divergence in Γ_k is cancelled by divergence in F_k. Indeed, if we consider the contribution of small wave-numbers in integral (2.5), we end up with the following expression:

$$F_k = 2\pi g^2\, N_k \int |T_{kk_1,kk_3}|^2\,\delta(\omega_{k_1} - \omega_{k_3})\, N_{k_1}N_{k_3}\, dk_1 dk_3 \simeq N_k\,\Gamma_k. \tag{6.17}$$

In negligence of γ_k, equation (4.1) is satisfied automatically.

The results obtained in this Chapter show that the four-wave nonlinear interaction is a very strong effect. Strong turbulence of near-surface air boundary layer makes the development of reliable theory of air-water interaction, including a well-justified analytical calculation of γ_k, an extremely difficult task. Field and laboratory measurements of γ_k are difficult either, and the scatter in determination of γ_k is of order of γ_k itself. Anyway, comparison of calculated above Γ_k with experimental data on γ_k shows that Γ_k surpasses γ_k at least in the order of magnitude. This fact is demonstrated on Figure 3, where experimental data are taken from [26].

As a result, we can make the conclusion that S_{nl} is the leading term in the balance equation (1.11) and that the rear face of the spectrum is describes by solution of equation (4.1), which has a rich family of solutions. In particulary, this equation describes the angular spreading.

On Figure 4 we demonstrate that for the nonlinear interaction term $S_{nl} = F_k - \Gamma_k\,N_k$ the magnitudes of constituents F_k and $\Gamma_k\,N_k$ essentially exceed their difference. They are one order higher than the magnitude of S_{nl}!

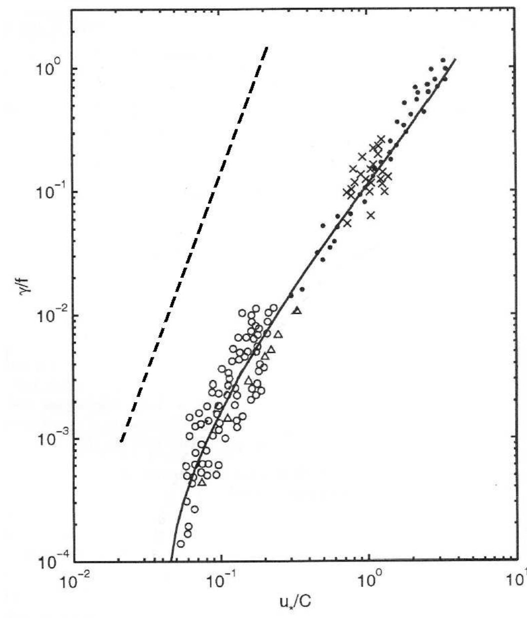

Figure 3: Comparison of experimental data for the wind-induced growth rate $2\pi\,\gamma_{in}(\omega)/\omega$ taken from [26] and the damping due to four-wave interactions $2\pi\,\Gamma(\omega)/\omega$, calculated for narrow in angle spectrum at $\mu \simeq 0.05$ using Eq. 6.11 (dashed line)

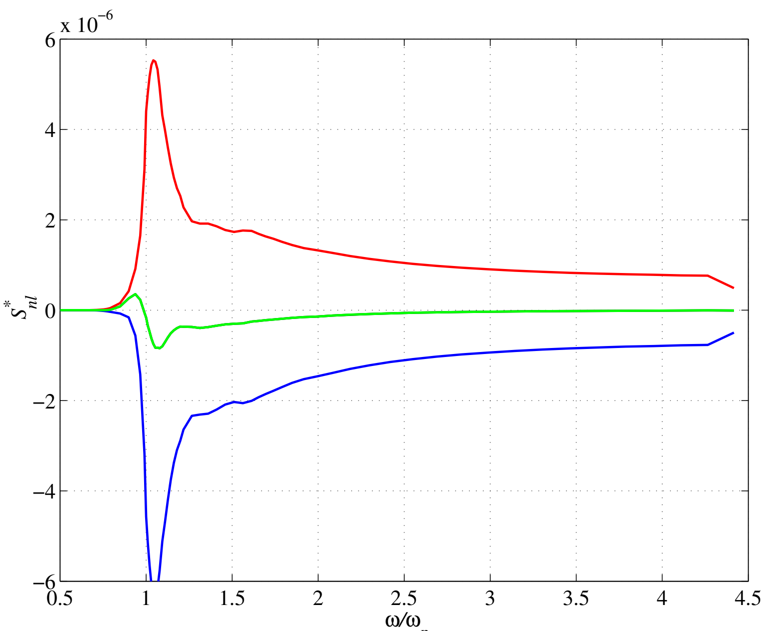

Figure 4: Split of nonlinear interaction term S_{nl} (central curve) into F_k (upper curve) and $\Gamma_k\,N_k$ (lower curve)

The dominance of S_{nl} was not apparent until now for two reasons. First, it is not correct to compare S_{nl} and S_{in}; instead one should compare Γ_k and γ_k. Second, the widely accepted models for S_{dis} essentially overestimate dissipation due to white capping. As a result, the dominance of S_{nl} is masked. We offer an alternative model for S_{dis}, which will be published in forthcoming article [27]. Preliminary results obtained in this direction were reported on ICNAAM-2009, Crete, Rethimno, September 2009 [28].

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