

FOUR-WAVE RESONANT INTERACTIONS IN THE FLAT BOTTOM BOUSSINESQ EQUATIONS

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MOTIVATION

WAVE MODELING IN SHALLOW WATER

Common thinking:

a) **FOUR-WAVE** RESONANT INTERACTIONS IN DEEP /INTERMEDIATE WATER

b) **THREE -WAVE** INTERACTIONS (NEVER RESONANT) IN SHALLOW WATER

i) computation of triplets

ii) deterministic modeling (Boussinesq or higher order)

iii) evolution equations for spectra and bi-spectra based on Boussinesq-like equations

APPARENTLY A “PHASE TRANSITION” OCCURS BETWEEN THE DEEP AND SHALLOW WATER!

i) Are the four-wave interactions relevant in shallow water?

ii) Is there a relation between the Boussinesq equations and the four-wave resonant interactions?

iii) Does an irreversible transfer of energy occur in flat-bottom shallow water waves?

METHODOLOGY

Theoretical analysis of the oldest, flat-bottom shallow water model:

THE BOUSSINESQ EQUATIONS

$$\eta_t + \nabla \cdot [(\eta + h)\mathbf{u}] = 0$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + g \nabla \eta - \frac{h^2}{3} \nabla \nabla \cdot \mathbf{u}_t = 0$$

SUMMARY OF THE RESULTS

i) A FOUR-WAVE HASSELMANN-LIKE EQUATION (BOUSSINESQ KINETIC EQUATION or BOUSSINESQ S_{NL}) HAS BEEN DERIVED FROM THE BOUSSINESQ EQUATIONS

ii) BOUSSINESQ KINETIC EQUATION CORRESPONDS TO THE ARBITRARY DEPTH HASSELMANN EQUATION WHEN THE LIMIT OF SMALL DEPTH IS PROPERLY TAKEN

iii) THE ESTIMATED TIME SCALE OF THE FOUR-WAVE NONLINEAR INTERACTIONS IN SHALLOW WATER IS

$$\tau_{\text{NL}} \sim \frac{1}{\omega} \left(\frac{kh}{\varepsilon} \right)^4 \sim \frac{1}{\omega} \left(\frac{1}{a/h} \right)^4$$

iv) DIRECT ENERGY CASCADE SOLUTIONS ARE FOUND TO BE OF THE FORM OF $E(k) \sim k^{-4/3}$ (CONSISTENT WITH ZAKHAROV EJMB/FLUIDS 1999)

- v) RESULTS ON WAVE SPECTRA CASCADE ARE CONFIRMED BY DIRECT NUMERICAL COMPUTATIONS OF THE BOUSSINESQ EQUATIONS.
- vi) EXPERIMENTAL EVIDENCE OF POWER LAW FOR INFROGRAVITY WAVES (see also Smith and Vincent 2003, Kaihatu et al. 2007 for shoaling waves)

References:

- Janssen and M.O., J. Phys. Oceanography (2007)
Zakharov, Europ. J. Mechanics B/Fluids (1999)
Zakharov, in Nonlinear Waves and Weak Turbulence (1998)

OUTLINE

- **Derivation of the Boussinesq kinetic equation**
- **Discussion of the Boussinesq kinetic equation and its relation to the Hasselmann equation**
- **Power law solutions of the Boussinesq kinetic equation**
- **Numerical simulations**
- **Some experimental data**

Derivation of the Boussinesq kinetic equations

$$\eta_t + \nabla \cdot [(\eta + h)\mathbf{u}] = 0$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + g \nabla \eta - \frac{h^2}{3} \nabla \nabla \cdot \mathbf{u}_t = 0$$

i) WRITE THE EQUATION IN FOURIER SPACE AND INTRODUCE THE COMPLEX AMPLITUDE

$$a(\mathbf{k}, t) = \left(\frac{g}{2\omega_k} \right)^{1/2} \eta(\mathbf{k}, t) + i\gamma_k \left(\frac{\omega_k}{2g} \right)^{1/2} \phi(\mathbf{k}, t)$$

with $\omega(k) = \sqrt{gh} \frac{k}{(1 + k^2 h^2 / 3)^{1/2}} \quad \gamma(k) = 1 + k^2 h^2 / 3$

ii) USE OF THE MULTIPLE SCALE EXPANSION:

- Introduce a slow time scale: $\tau = \varepsilon^2 t$
- Look for a solution of the form:

$$a_k(t, \tau) = b_k(t, \tau) + \varepsilon b_k^{(1)}(t, \tau) + \varepsilon^2 b_k^{(2)}(t, \tau) + \dots$$

$$\frac{\partial b_0}{\partial t} + i\omega_0 b_0 = -i\varepsilon^2 \int T_{0123}^{(B)} b_1^* b_2 b_3 \delta(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_{123}$$

Narrow band limits of the Boussinesq-Zakharov equation

- Long crested case (one dimensional propagation):

Defocusing Nonlinear Schroedinger equation in shallow water

- Weakly two-dimensional case:

Shallow water limit of the Davey-Stewartson

BOTH EQUATIONS ARE INTEGRABLE!!

NO ENERGY TRANSFER!!

iii) STATISTICAL DESCRIPTION OF THE SHALLOW WATER ZAKHAROV EQUATION

Hypothesis:

i) Homogeneity

ii) Quasi Gaussian approximation

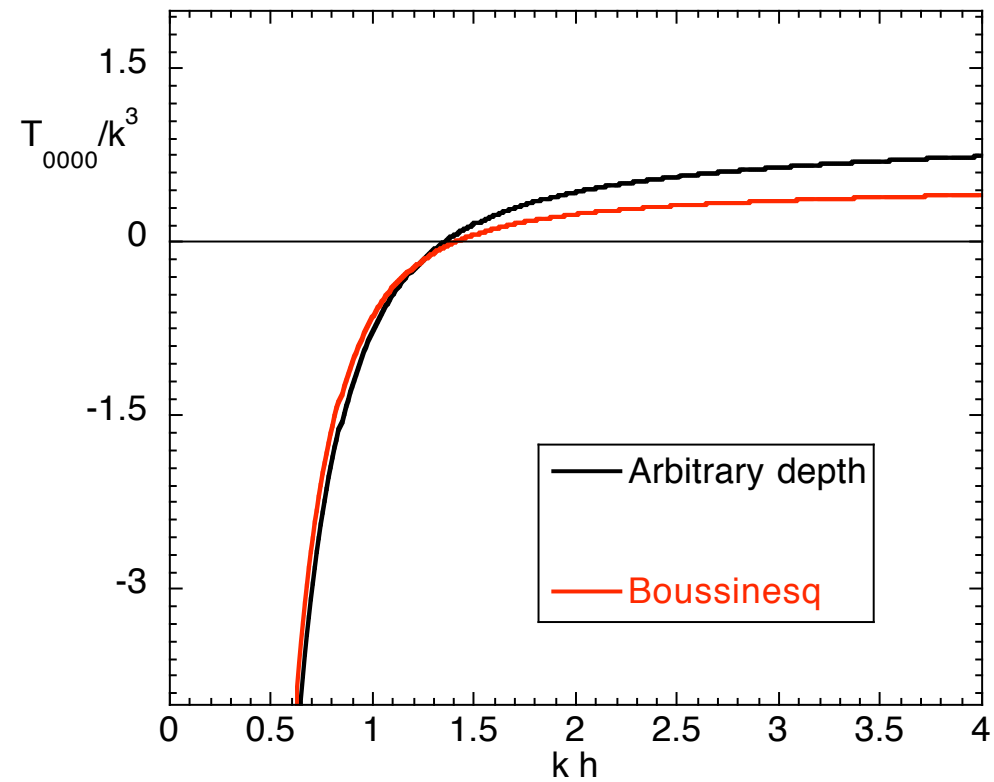
$$\frac{\partial N_0}{\partial t} = \int (T_{0123}^{(B)})^2 N_0 N_1 N_2 N_3 \left(\frac{1}{N_0} + \frac{1}{N_1} - \frac{1}{N_2} - \frac{1}{N_3} \right) \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3) \delta(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_{123}$$

where

$$\langle a(\mathbf{k}_i, t) a^*(\mathbf{k}_j, t) \rangle = N(\mathbf{k}_i, t) \delta(\mathbf{k}_i - \mathbf{k}_j)$$

Relation with the Hasselmann equation

$$T_{0123}^{(B)} = \lim_{kh \rightarrow 0} T_{0123}^{(ad)}$$



NONLINEAR TIME SCALE

$$\frac{\partial N_0}{\partial t} = \int (T_{0123}^{(B)})^2 N_0 N_1 N_2 N_3 \left(\frac{1}{N_0} + \frac{1}{N_1} - \frac{1}{N_2} - \frac{1}{N_3} \right) \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3) \delta(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_{123}$$

$$1/\tau_{NL} \sim (T^{(B)})^2 \omega^{-1} N^2 k^4$$

with $T^{(B)} \sim k^2/h$

$$\tau_{NL} \sim \frac{1}{\omega} \left(\frac{kh}{\varepsilon} \right)^4 \sim \frac{1}{\omega} \left(\frac{1}{a/h} \right)^4$$

Power law solutions of the Boussinesq kinetic equation

Hypothesis: existence of an equilibrium range in Fourier space where there is a constant flux of energy, independent of k

Dimensional analysis of the kinetic equation

$$\Delta N / \Delta t \sim T^{(B)2} \omega^{-1} N^3 k^4 \text{ with } T^{(B)} \sim k^2 / h$$

Flux of energy Π :

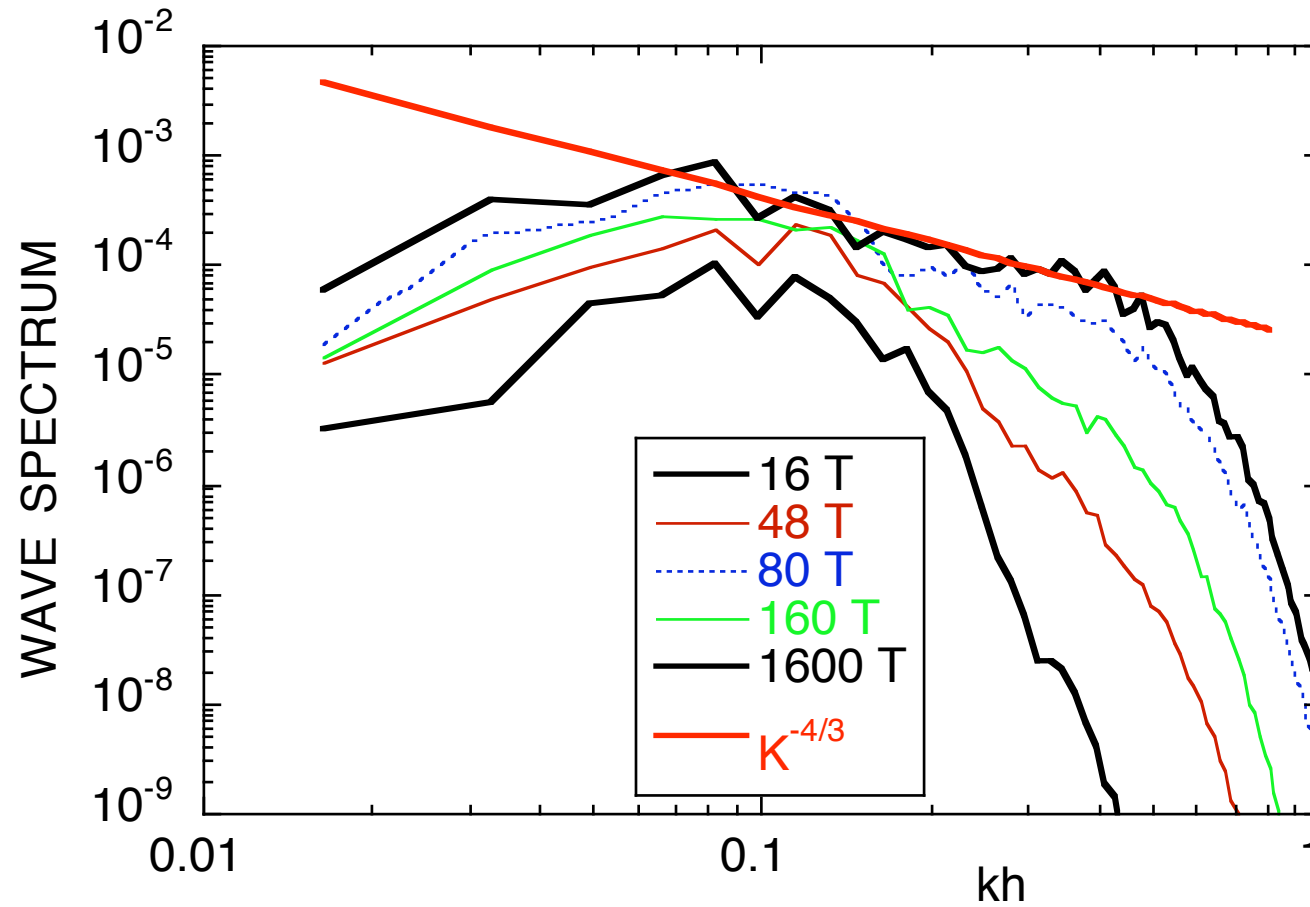
$$\Pi \sim k^2 \Delta E / \Delta t \sim N^3 k^{10}$$

must be independent of k , therefore

$$N \sim k^{-10/3} \text{ or } E \sim k^{-4/3}$$

(in agreement with Zakharov 1999)

Numerical simulations of Boussinesq equations



EFFECTS OF REGULAR DISCRETIZATION: LACK OF EXACT RESONANCES

Example:

$$h=1 \text{ m}, k_p=0.2 \text{ m}^{-1}$$

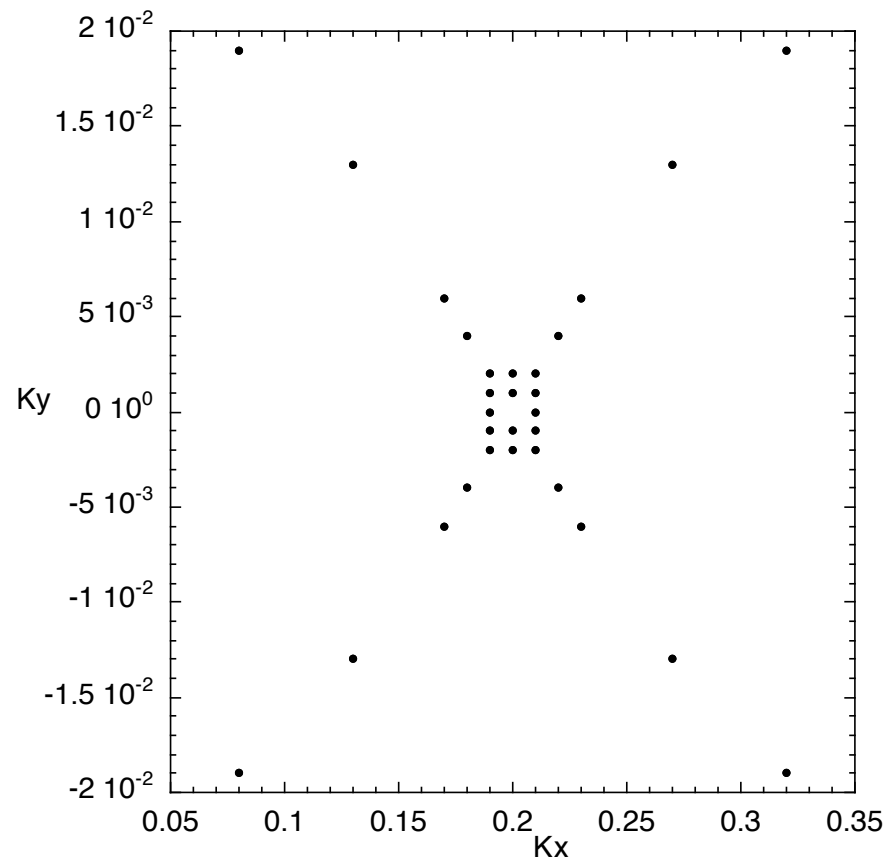
$$\Delta k_x=0.01 \text{ m}^{-1}$$

$$\Delta k_y=0.001 \text{ m}^{-1}$$

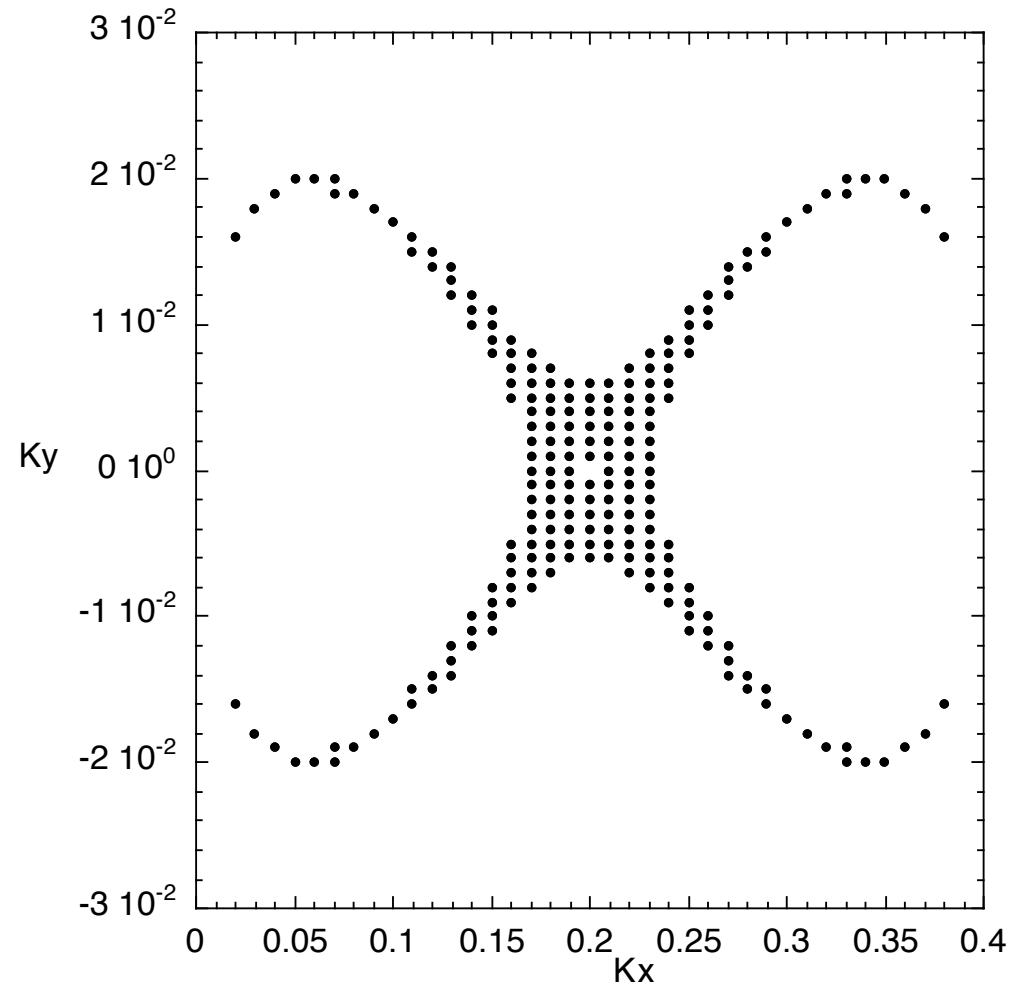
$$k_1 = k_p, k_2 = k_p$$

**IN A 256x256 GRID THERE ARE NOT EXACT
RESONANCES!**

$$|\omega_1 + \omega_2 - \omega_3 - \omega_4| < 10^{-4} \omega_1$$



$$|\omega_1 + \omega_2 - \omega_3 - \omega_4| < 10^{-3} \omega_1$$



Some comparison with experimental data (from Smith and Vincent JGR 2003)

SMITH AND VINCENT: EQUILIBRIUM RANGES IN SURF ZONE SPECTRA

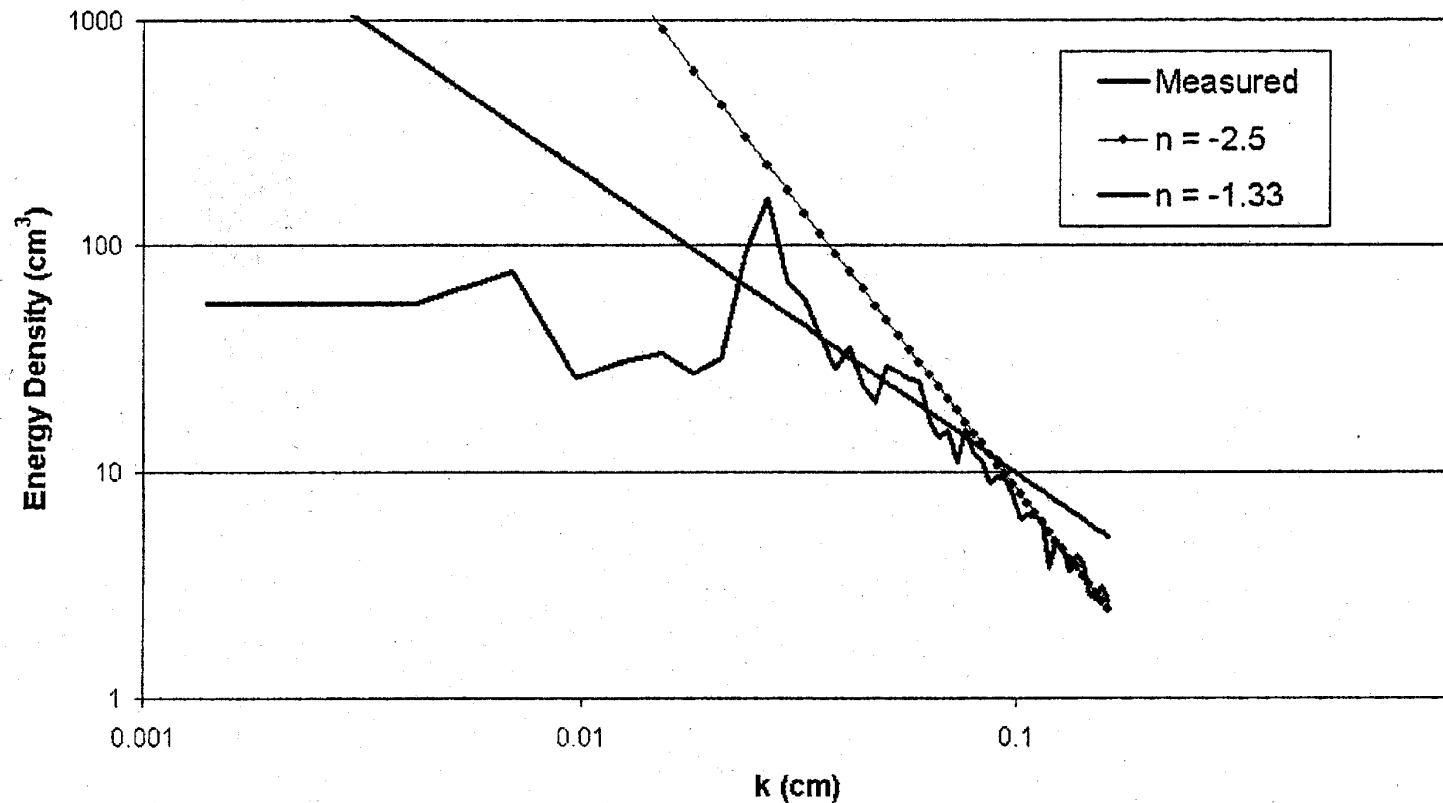
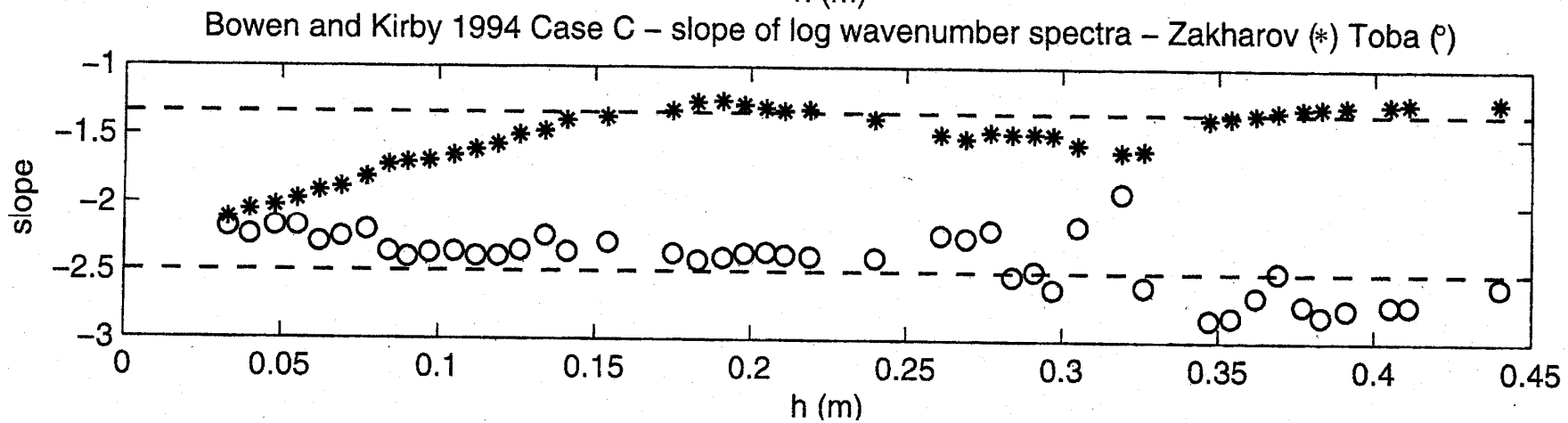
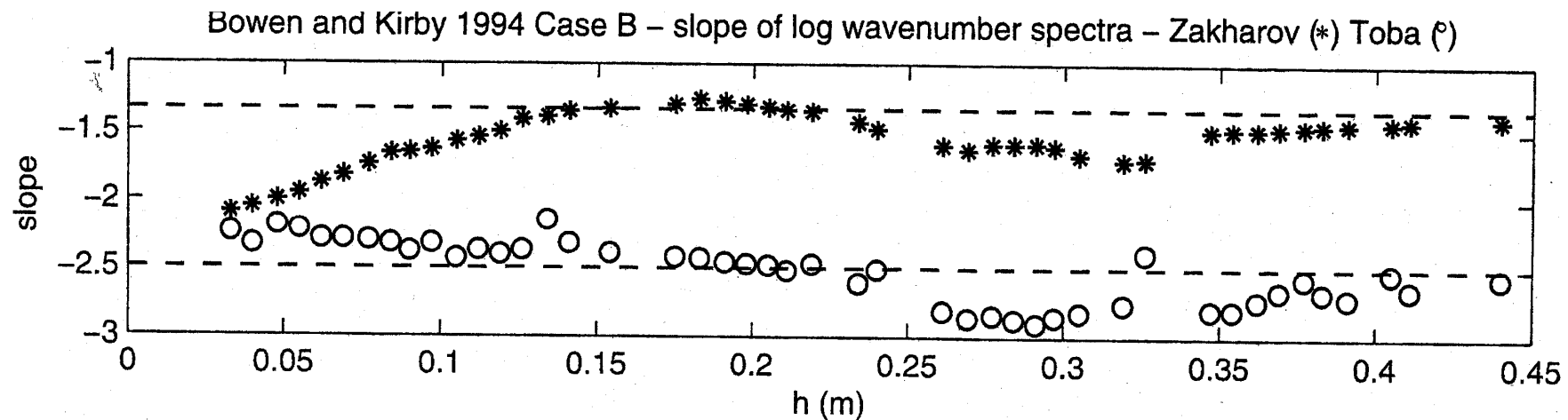


Figure 3. Example of Toba equilibrium range ($n = -5/2$) fit to lab data ($d = 18.3$ cm).

Some comparison with experimental data (from Kaihatu et al. JGR 2007)



From Resio and Long (JGR 2006)

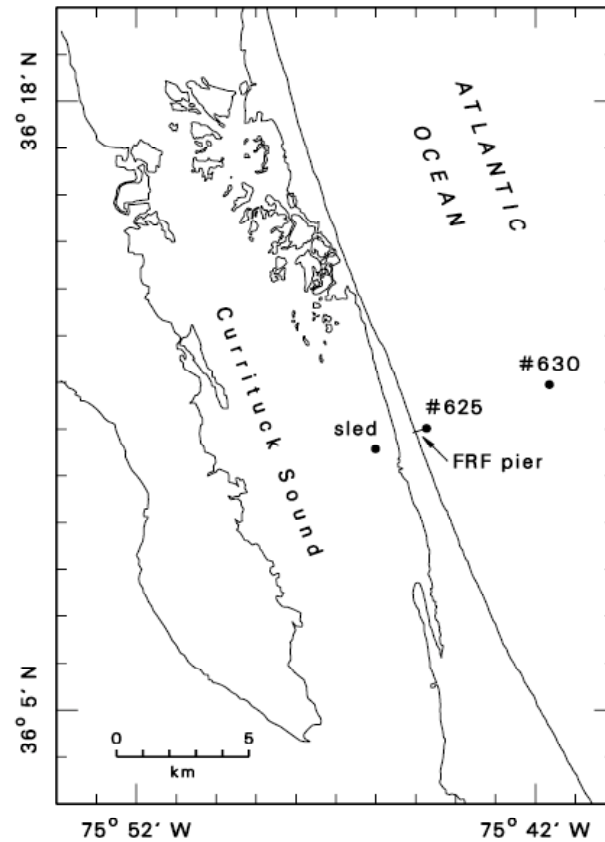
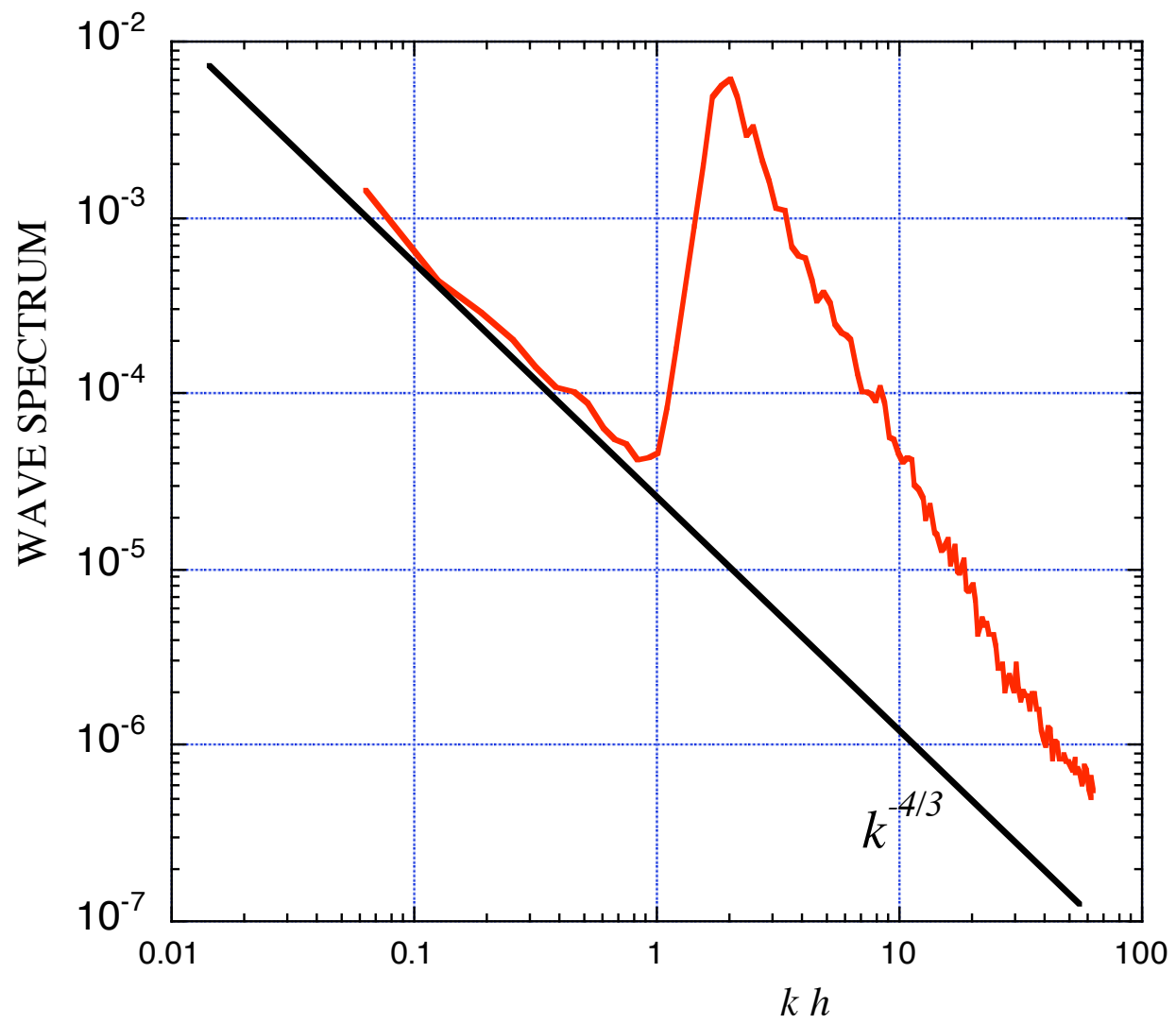


Figure 2. Site map showing the boundaries of Currituck Sound nearby the instrument platform (solid circle labeled “sled”). The sound is isolated from the Atlantic Ocean to the east by the barrier island of North Carolina’s Outer Banks. At the nearest points, the sled is about 1 km from the east side of the sound and about 6 km from the west side. A broad channel of about 2.5-m depth runs along the long axis of the sound in the vicinity of the sled. This channel narrows considerably to the north. The opening to the south, representing an azimuthal arc of about 20 deg from the sled location, leads to the larger, deeper Albemarle Sound.



THREE WAVE INTERACTIONS CAN BE ASYMPTOTICALLY RESONANT

$$k_1 + k_2 = k_3$$

$$\omega_1 + \omega_2 \sim \omega_3$$

If k_1 is close to k_2 then

$$\omega_1 + \omega_2 - \omega_3 \sim c_0 (k_1 - k_2) (kh)^2 \sim 0$$

CONCLUSIONS

- The Boussinesq kinetic equation is equivalent to the shallow water limit of the Hasselmann equation
- Four-wave resonant interaction can be relevant in shallow water
- Power law solutions of the Boussinesq kinetic have been confirmed by numerical simulations
- There is some evidence that this solution can be observed in the surf-zone
- Infragravity waves in Currituck Sound show also power law $k^{-4/3}$